Math 3345
Fundamentals of Higher Mathematics

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Course Info

HW25 Due Wednesday, April 9, 2014

- Section 11 Exercises 22a, 22b, 23ab

Warm-up Problems

Problem 1

Solution to Section 11 Exercise 11:

Let f, g and h be functions. Let

$$B = \{x \in \mathrm{Dom}(f) \mid f(x) \in \mathrm{Dom}(g) \text{ and } g(f(x)) \in \mathrm{Dom}(h)\}$$
$$C = \mathrm{Dom}(h \circ (g \circ f))$$

Then $B = C$ and for all $x \in B$ we have

$$h(g(f(x))) = (h \circ (g \circ f))(x)$$

Infinite sequence

Definition 2 (Infinite sequence)

Let A be a set. An **infinite sequence** in A is a function $f : \mathbf{N} \to A$.

Notation

Let f be an infinite sequence in the set A and for each $n \in \mathbf{N}$ let $a_n = f(n)$. We usually denote the sequence given by the function f as $(a_n)_{n \in \mathbf{N}}$ or sometimes

$$a_1, a_2, a_3, a_4, \cdots$$

if the pattern for the terms of the sequence is clear.

Example 3 (Infinite sequences)

1. The sequence $\left(\frac{n+1}{n}\right)_{n\in\mathbf{N}}$ is a sequence in $\mathbf{Q}$ and has first few terms
$$\tfrac{2}{1}, \tfrac{3}{2}, \tfrac{4}{3}, \tfrac{5}{4}, \cdots$$
2. The sequence $(0)_{n\in\mathbf{N}}$ is a sequence in $\mathbf{R}$ and has first few terms
$$0, 0, 0, 0, \cdots$$
3. The sequence $\big([n, n+1)\big)_{n\in\mathbf{N}}$ is a sequence in $\mathcal{P}(\mathbf{R})$ and has first few terms
$$[1,2), [2,3), [3,4), [4,5), \cdots$$
Notice that each term of this sequence is a set.
4. For each $n \in \mathbf{N}$ let $f_n(x) = \sin(nx)$ then $(f_n)_{n\in\mathbf{N}}$ is a sequence in the set of functions from $\mathbf{R}$ to $\mathbf{R}$. Notice that each term of this sequence is itself a function.

Families of Sets

Definition 4 (Family of sets)

Let A be a set. Suppose that for each $a \in A$ there is a set B_a then $(B_a)_{a\in A}$ is a **family of sets indexed by** A.

Definition 5 (Intersection and union of a family of sets)

Let $(B_a)_{a\in A}$ be family of sets indexed by the set A. Then the **union of the family of sets** $(B_a)_{a\in A}$ is the set
$$\bigcup_{a\in A} B_a = \{x \mid (\exists a \in A) \text{ such that } x \in B_a\}$$
and the **intersection of the family of sets** $(B_a)_{a\in A}$ is the set
$$\bigcap_{a\in A} B_a = \{x \mid (\forall a \in A)\, x \in B_a\}$$

Note that a family of sets indexed by $\mathbf{N}$ is just a sequence of sets.

Families of Sets

Example 6 (Families of sets)

1. The family of sets $(B_n)_{n\in\mathbf{N}}$ where $B_n = [0, n)$ has union
$$\bigcup_{n\in\mathbf{N}} B_n = [0, \infty)$$
and intersection
$$\bigcap_{n\in\mathbf{N}} B_n = [0, 1)$$
2. The family of sets $(C_x)_{x\in\mathbf{R}}$ where $C_x = (x, \infty)$ has union
$$\bigcup_{x\in\mathbf{R}} C_x = \mathbf{R}$$
and intersection
$$\bigcap_{x\in\mathbf{R}} C_x = \varnothing$$

Sets of Functions

Definition 7

Let A and B be sets. The **set of all functions from A to B** is
$$B^A = \{f \mid f : A \to B\}.$$

Example 8

1. Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Then
$$B^A = \Big\{\{(1,3),(2,3)\},\{(1,3),(2,4)\},\{(1,4),(2,3)\},\{(1,4),(2,4)\}\Big\}$$
Note that B^A has 4 elements each of which is a function.
2. $$\varnothing^{\{1,2\}} = \{f \mid f : \{1,2\} \to \varnothing\} = \varnothing$$
3. $$\{1,2\}^{\varnothing} = \Big\{f \mid f : \varnothing \to \{1,2\}\Big\} = \{\varnothing\}$$