Math 3345
Fundamentals of Higher Mathematics

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Course Info

HW25 Due Wednesday, April 9, 2014

► Section 11 Exercises 22a, 22b, 23ab
Warm-up Problems

Problem 1

Solution to Section 11 Exercise 11:

Let $f$, $g$ and $h$ be functions. Let

$$B = \{x \in \text{Dom}(f) \mid f(x) \in \text{Dom}(g) \text{ and } g(f(x)) \in \text{Dom}(h)\}$$

$$C = \text{Dom}(h \circ (g \circ f))$$

Then $B = C$ and for all $x \in B$ we have

$$h(g(f(x))) = (h \circ (g \circ f))(x)$$

Infinite sequence

Definition 2 (Infinite sequence)

Let $A$ be a set. An **infinite sequence** in $A$ is a function $f : \mathbb{N} \to A$.

Notation

Let $f$ be an infinite sequence in the set $A$ and for each $n \in \mathbb{N}$ let $a_n = f(n)$. We usually denote the sequence given by the function $f$ as $(a_n)_{n \in \mathbb{N}}$ or sometimes

$$a_1, a_2, a_3, a_4, \cdots$$

if the pattern for the terms of the sequence is clear.
Example 3 (Infinite sequences)

1. The sequence $(\frac{n+1}{n})_{n \in \mathbb{N}}$ is a sequence in $\mathbb{Q}$ and has first few terms:
   
   \[
   \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots
   \]

2. The sequence $(0)_{n \in \mathbb{N}}$ is a sequence in $\mathbb{R}$ and has first few terms:
   
   \[
   0, 0, 0, 0, \ldots
   \]

3. The sequence $([n, n+1])_{n \in \mathbb{N}}$ is a sequence in $\mathcal{P}(\mathbb{R})$ and has first few terms:
   
   \[
   [1, 2), [2, 3), [3, 4), [4, 5), \ldots
   \]
   
   Notice that each term of this sequence is a set.

4. For each $n \in \mathbb{N}$ let $f_n(x) = \sin(nx)$ then $(f_n)_{n \in \mathbb{N}}$ is a sequence in the set of functions from $\mathbb{R}$ to $\mathbb{R}$. Notice that each term of this sequence is itself a function.

Families of Sets

Definition 4 (Family of sets)

Let $A$ be a set. Suppose that for each $a \in A$ there is a set $B_a$ then $(B_a)_{a \in A}$ is a family of sets indexed by $A$.

Definition 5 (Intersection and union of a family of sets)

Let $(B_a)_{a \in A}$ be family of sets indexed by the set $A$. Then the union of the family of sets $(B_a)_{a \in A}$ is the set:

\[
\bigcup_{a \in A} B_a = \{x \mid (\exists a \in A) \text{ such that } x \in B_a\}
\]

and the intersection of the family of sets $(B_a)_{a \in A}$ is the set:

\[
\bigcap_{a \in A} B_a = \{x \mid (\forall a \in A) x \in B_a\}
\]

Note that a family of sets indexed by $\mathbb{N}$ is just a sequence of sets.
Families of Sets

Example 6 (Families of sets)

1. The family of sets \( (B_n)_{n \in \mathbb{N}} \) where \( B_n = [0, n) \) has union

\[
\bigcup_{n \in \mathbb{N}} B_n = [0, \infty)
\]

and intersection

\[
\bigcap_{n \in \mathbb{N}} B_n = [0, 1)
\]

2. The family of sets \( (C_x)_{x \in \mathbb{R}} \) where \( C_x = (x, \infty) \) has union

\[
\bigcup_{x \in \mathbb{R}} C_x = \mathbb{R}
\]

and intersection

\[
\bigcap_{x \in \mathbb{R}} C_x = \emptyset
\]

Sets of Functions

Definition 7

Let \( A \) and \( B \) be sets. The set of all functions from \( A \) to \( B \) is

\[
B^A = \{ f \mid f : A \to B \}.
\]

Example 8

1. Let \( A = \{1, 2\} \) and \( B = \{3, 4\} \). Then

\[
B^A = \{ \{(1, 3), (2, 3)\}, \{(1, 3), (2, 4)\}, \{(1, 4), (2, 3)\}, \{(1, 4), (2, 4)\} \}
\]

Note that \( B^A \) has 4 elements each of which is a function.

2. \( \emptyset^{\{1,2\}} = \{ f \mid f : \{1, 2\} \to \emptyset \} = \emptyset \)

3. \( \{1, 2\}^\emptyset = \{ f \mid f : \emptyset \to \{1, 2\} \} = \{\emptyset\} \)