Math 3345
Fundamentals of Higher Mathematics

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Comparing the sizes of sets

Definition 1 (Restricting the domain of a function)

Let $f : A \to B$. If $C \subset A$ then the **restriction of** f **to** C is the function $f_| C$ such that

$$f|_C : C \to B$$

and for all $c \in C$ $f|_C(c) = f(c)$.

Lemma 2

If $f : A \to B$ is an injection and $C \subset A$ then $f|_C : C \to B$ is an injection.

Lemma 3

If $f : A \to B$ is a bijection and $C \subset A$ then $f|_C : C \to f[C]$ is a bijection.

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Lemma 4

Let A and B be sets and suppose A is equinumerous with B. If $s \notin A$ and $t \notin B$ then $A \cup \{s\}$ is equinumerous with $B \cup \{t\}$.

Proof.

Suppose $|A| = |B|$, $s \notin A$ and $t \notin B$. Then we have a bijection $g : A \to B$ and a bijection $h : \{s\} \to \{t\}$ where $h(s) = t$. The sets A and $\{s\}$ are disjoint and the sets B and $\{t\}$ are disjoint so we may apply Problem 11.22.c to conclude that $\varphi : A \cup \{s\} \to B \cup \{t\}$ is a bijection. Hence $A \cup \{s\}$ is equinumerous with $B \cup \{t\}$. □

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Lemma 5

Let A and B be sets and suppose A is equinumerous with B. If $s \in A$ and $t \in B$ then $A \setminus \{s\}$ is equinumerous with $B \setminus \{t\}$.

Proof.

Suppose $|A| = |B|$, $s \in A$ and $t \in B$. Then we have a bijection $f : A \to B$. Either $f(s) = t$ or $f(s) \neq t$.

Case I: Suppose $f(s) = t$.

Then f is an injection so by Theorem 12.26

$$f[A \setminus \{s\}] = f[A] \setminus f[\{s\}] = B \setminus \{t\}$$

By Lemma 3 above $f|_{A \setminus \{s\}} : A \setminus \{s\} \to B \setminus \{t\}$ is a bijection. Hence from The sets A and $\{s\}$ are disjoint and the sets B and $\{t\}$ are disjoint so we may apply Problem 11.22.c to conclude that $\varphi : A \cup \{s\} \to B \cup \{t\}$ is a bijection. Hence $A \setminus \{s\}$ is equinumerous with $B \setminus \{t\}$.

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Proof of Lemma 5 *(continued)*.

Case II: Suppose $f(s) \neq t$.
Define $h : B \to B$ to be the function

$$h(b) = \begin{cases} f(s), & b = t \\ t, & b = f(s) \\ b, & b \notin \{t, f(s)\} \end{cases}$$

$h|_{B \setminus \{t,f(s)\}} = \mathrm{id}_{B \setminus \{t,f(s)\}}$ and is therefore a bijection. $h|_{\{t,f(s)\}}$ is a bijection applying Problem 11.22.c we may conclude that $h : B \to B$ is a bijection. The composition of bijections is a bijection hence $h \circ f : A \to B$ is a bijection and $h \circ f(s) = h(f(s)) = t$. Thus we may apply Case I with the bijection $h \circ f$ to conclude that $A \setminus \{s\}$ is equinumerous with $B \setminus \{t\}$. □

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Theorem 6

A finite set cannot be equinumerous with a proper subset of itself.

Proof.

We will proceed by induction. Let $P(n)$ be the statement

$$\text{If } |A| = n \text{ then no proper subset } B \subset A \text{ satisfies } |A| = |B|.$$

Base Case: Suppose $|A| = 0$.
Then there is a bijection $f : \varnothing \to A$. Hence $A = \varnothing$. If $B \subset A$ then $B \subset \varnothing$ so $B = \varnothing = A$. Thus A has no proper subsets. If follows that $P(0)$ is vacuously true.

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Proof of Theorem 6 *(continued).*

Inductive Step: Suppose $n \in \omega$ and $P(n)$ is true.
Suppose $|A| = n+1$ and $B \subset A$ satisfies $|B| = |A|$. Then $|A| = n+1 \geq 1$ thus A is not empty. Hence B is not empty. Thus there exists $t \in B$. Also $t \in A$ since $B \subset A$. $A \setminus \{t\}$ is equinumerous with $\{1, 2, \cdots, n+1\} \setminus \{n+1\}$ by Lemma 5. Hence $|A \setminus \{t\}| = n$. Similarly $B \setminus \{t\}$ is equinumerous with $\{1, 2, \cdots, n+1\} \setminus \{n+1\}$ by Lemma 5. Hence $|B \setminus \{t\}| = n$. $B \setminus \{t\} \subset A \setminus \{t\}$. Thus by inductive assumption $B \setminus \{t\} = A \setminus \{t\}$ so

$$B = \{t\} \cup (B \setminus \{t\}) = \{t\} \cup (A \setminus \{t\}) = A$$

Thus B is not a proper subset of A. Hence $P(n+1)$ is true. □