Math 3345
Fundamentals of Higher Mathematics

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Warm-up Problems

Problem 1

Solution to Quiz 4

Infinite Sets

Definition 2

1. A set B is **finite** if there is $n \in \omega$ such that B has n elements.
2. A set is **infinite** if it is not finite.
3. A set is **countable** if it is finite or equinumerous with $\mathbf{N}$.
4. A set is **denumerable** (or **countably infinite**) if it is equinumerous with $\mathbf{N}$.
5. A set is **uncountable** if it is not countable.

Denumerable Sets

Proving that a set is infinite can be done using the contrapositive of Theorem 6 from Lecture 32.

Theorem 3

A set A is infinite if there is an injection $f : A \to A$ which is not a surjection.

Proof.

Suppose we have an injection $f : A \to A$ which is not a surjection. Then $f : A \to f[A]$ is a bijection so A is equinumerous with the set $f[A] \subset A$. Since f is not a surjection $f[A] \neq A$. Hence A is equinumerous to a proper subset of itself. Thus by Theorem 6 from Lecture 32 A cannot be finite. Thus A is infinite. □

Denumerable Sets

Theorem 4

N *is an infinite set.*

Proof.

Let the function f satisfy $f : \mathbf{N} \to \mathbf{N}$ and for all $n \in \mathbf{N}$ let $f(n) = n + 1$. Then f is an injection but not a surjection (can you show this?). Thus by Theorem 3 above **N** must be infinte. □

Denumerable sets

Example 5 (**N** is denumerable.)

From Theorem 4 above **N** is not finite. $\mathrm{id}_{\mathbf{N}} : \mathbf{N} \to \mathbf{N}$ is a bijection thus **N** is equinumerous with **N**.

Example 6 (**Z** is denumerable.)

Let $g : \mathbf{N} \to \mathbf{Z}$ be the function

$$g(n) = \begin{cases} \frac{n}{2}, & n \text{ is even} \\ \frac{n-1}{2}, & n \text{ is odd} \end{cases}$$

Then g is a bijection so **Z** is denumerable.

Denumerable sets

Example 7 ($\mathbf{Z} \times \mathbf{Z}$ is denumerable.)

Let $h : \mathbf{N} \to \mathbf{Z} \times \mathbf{Z}$ be the "spiral function" with

$$h(1) = (0,0) \quad h(2) = (1,0) \quad h(3) = (1,1) \quad h(4) = (0,1) \quad \textit{etc.}$$

Then h is a bijection so $\mathbf{Z} \times \mathbf{Z}$ is denumerable.

Example 8 ($\mathbf{Q}$ is denumerable.)

Let $s : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Q}$ be the function

$$s(a,b) = \begin{cases} \frac{a}{b}, & b \neq 0 \\ 0, & b = 0 \end{cases}$$

Then s is a surjection and h from Example 7 is bijection so $s \circ h : \mathbf{N} \to \mathbf{Q}$ is a surjection. Define $r : \mathbf{N} \to \mathbf{Q}$ resurvively by setting $r(1) = s \circ h(1)$ and for all $n \in \mathbf{N}$ $r(n+1) = s \circ h(k)$ where
$k = \min\{x \in \mathbf{N} \mid s \circ n(x) \neq r(y)$ for all $y \in \mathbf{N}$ such that $1 \leq y \leq n\}$.