Math 3345 – April 21, 2014

Practice Final Exam

Instructions:

• You will have 105 min for this exam.
• Cross out all scratchwork from you final answer.
• You may use any theorems from class or the book unless instructed not to.
• Indicate the starting point for your proofs clearly.
• List all of your assumptions at the beginning of your proofs.
• You may use any proof techniques (conditional proof, proof by contraposition, induction, complete induction ...) you like.

1. (10 points) Decide if the following statements are TRUE or FALSE and circle your answer. You do NOT need to justify your answers.

   T    F   (a) If A and B are sets and A ⊂ B then \( \mathcal{P}(A) \subset \mathcal{P}(B) \).
   T    F   (b) If the set A is equinumerous with the set B then A \( \setminus \) B = \( \varnothing \).
   T    F   (c) If P and Q are statements then \( (P \Rightarrow Q) \Rightarrow Q \) is a tautology.
   T    F   (d) If sets A and B are disjoint then \( \mathcal{P}(A) \) and \( \mathcal{P}(B) \) are disjoint.
   T    F   (e) Let \( f : (0,1] \rightarrow (0,1) \) be the function

   \[
   f(x) = \begin{cases} 
   x, & (\forall n \in \mathbb{N}) x \neq \frac{1}{n} \\
   \frac{1}{n+4}, & (\exists n \in \mathbb{N}) x = \frac{1}{n} 
   \end{cases}.
   \]

   Then \( f \) is a surjection.

   T    F   (f) The identity function for the empty set id_\( \varnothing \) is the empty function.

   T    F   (g) If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a bijection then \( f = \text{id}_\mathbb{R} \).

   T    F   (h) If the set A has a proper subset B such that A is equinumerous to B then A must be infinite.

   T    F   (i) Suppose that

   • \( P(n) \) is a sentence depending on \( n \).
   • \( P(1) \) is true.
   • \( (\forall a, b \in \mathbb{N}) (P(a) \text{ and } P(b)) \Rightarrow P(ab) \).

   Then \((\forall n \in \mathbb{N}) P(n) \) is true.

   T    F   (j) For \( \alpha \in \mathbb{R} \) let \( B_\alpha = (\alpha, \infty) \times (-\infty, \alpha) \). Then

   \[ \bigcup_{\alpha \in \mathbb{R}} B_\alpha = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x > y \} \]
2. (10 points) Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**
   (a) Let \( A = \{1, 2, 3\} \). Give two different bijections \( f : A \to A \) and \( g : A \to A \).
   (b) Give an injection \( h : \mathbb{R} \to \mathcal{P}(\mathbb{R} \times \mathbb{R}) \).
   (c) Give two different elements of \( \mathcal{P}(\mathcal{P}(\{1, 2\})) \).
   (d) Give a sequence of sets \( (B_n)_{n \in \mathbb{N}} \) such that \( \bigcup_{n \in \mathbb{N}} B_n = \mathbb{Z} \) and \( \bigcap_{n \in \mathbb{N}} B_n = \emptyset \).
   (e) Give an element of \( \{(0, 1, 2)\}^\mathbb{R} \).
   (f) Give a function \( f : \mathbb{N} \to \mathbb{N} \) such that \( f = f^{-1} \).
   (g) Give two irrational numbers \( x, y \in \mathbb{R} \) such that \( (x + y) \) is rational.
   (h) Give an element of \( \mathbb{R}^3 \).
   (i) For each \( n \in \mathbb{N} \) let \( D_n = \{ k \in \mathbb{Z} \mid n \text{ divides } k \} \). Give an element of \( \bigcap_{n \in \mathbb{N}} D_n \).
   (j) Give a nonempty set \( B \subset \mathbb{Z} \) such that \( x, y \in B \) then \( \frac{x}{y} \in B \).
   (k) Give a function \( s : \mathbb{R} \to \mathbb{N} \) which is a surjection.
   (l) Give a function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f \circ f = f \) and \( f \) is not the identity function \( \text{id}_{\mathbb{R}} \).

3. (5 points) Using only the relevant definitions prove that if \( \mathcal{A} \) is a nonempty set of sets and \( \mathcal{B} \) is a set then
\[
\mathcal{B} \setminus \bigcap \mathcal{A} = \bigcup \{ \mathcal{B} \setminus A \mid A \in \mathcal{A} \}
\]

4. (5 points) Using only the relevant definitions prove that for any set \( A \) the power set \( \mathcal{P}(A) \) is equinumerous with the set \( \{0, 1\}^A \).

5. (5 points) **Without referring to any theorems from the text or lecture** prove that for all \( n \in \omega \)
\[
\sum_{k=n}^{2n} k = \frac{n(3n + 3)}{2}
\]

6. (5 points) Using only the relevant definitions and the following two facts prove that if \( x \) is rational and \( n \in \mathbb{Z} \) then if \( x^3 = n \) then \( x \) is an integer.

**FACT I:** If \( x \in \mathbb{Q} \) then there exist \( a \in \mathbb{Z} \) and \( b \in \mathbb{N} \) such that \( x = \frac{a}{b} \).

**FACT II:** If \( d, n_1, n_2, \ldots, n_k \in \mathbb{Z} \) and \( n = n_1 n_2 \cdots n_k \) and \( d \) divides \( n \) then there are \( d_1, d_2, \ldots, d_k \in \mathbb{Z} \) such that \( d_1 \) divides \( n_1, d_2 \) divides \( n_2, \ldots, d_k \) divides \( n_k \) and \( d = d_1 d_2 \cdots d_k \).

7. (5 points) **Using only the relevant definitions and the following fact** prove that if \( n \) and \( m \) are integers, \( n \) divides \( m \) and \( m \) divides \( n \) then \( n = \pm m \).

**FACT:** If \( x \in \mathbb{Z} \) and \( x \) divides 1 then \( x \in \{-1, 1\} \).

8. (5 points) Let \( \mathcal{S} \) and \( \mathcal{T} \) are sets and define a function
\[
f : \mathcal{P}(\mathcal{S}) \times \mathcal{P}(\mathcal{T}) \to \mathcal{P}(\mathcal{S} \cup \mathcal{T})
\]
by \( f(A, B) = A \cup B \) for all \( A \subseteq \mathcal{S} \) and \( B \subseteq \mathcal{T} \). Show that \( f \) is a surjection.

9. (5 points) Given \( n \in \{2, 3, 4, \cdots\} \) and \( k \in \{1, 2, \cdots, n - 1\} \) let
\[
A = \{1, 2, 3, \cdots, n\} \quad \text{and} \quad B = \{1, 2, \cdots, n - 1\}.
\]
Let
\[
M = \mathcal{P}_k(B) \cup \mathcal{P}_{k+1}(B).
\]
Show that there is a surjection \( f : \mathcal{P}(A) \to M \).