Math 3345 – April 21, 2014

Practice Final Exam

Instructions:

- You will have 105min for this exam.
- Cross out all scratchwork from you final answer.
- You may use any theorems from class or the book unless instructed not to.
- Indicate the starting point for your proofs clearly.
- List all of your assumptions at the beginning of your proofs.
- You may use **any proof techniques** (conditional proof, proof by contraposition, induction, complete induction ...) you like.

1. *(10 points)* Decide if the following statements are TRUE or FALSE and circle your answer. **You do NOT need to justify your answers.**

T **F** (a) If A and B are sets and $A \subset B$ then $\mathcal{P}(A) \subset \mathcal{P}(B)$.

T **F** (b) If the set A is equinumerous with the set B then $A \setminus B = \varnothing$

T **F** (c) If P and Q are statements then $(P \Rightarrow Q) \Rightarrow Q$ is a tautology.

T **F** (d) If sets A and B are disjoint then $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are disjoint.

T **F** (e) Let $f : (0,1] \to (0,1)$ be the function

$$f(x) = \begin{cases} x, & (\forall n \in \mathbf{N})\, x \neq \frac{1}{n} \\ \frac{1}{n+4}, & (\exists n \in \mathbf{N})\, x = \frac{1}{n} \end{cases}.$$

Then f is a surjection.

T **F** (f) The identity function for the empty set $\mathrm{id}_\varnothing$ is the empty function.

T **F** (g) If $f : \mathbf{R} \to \mathbf{R}$ is a bijection then $f = \mathrm{id}_{\mathbf{R}}$.

T **F** (h) If the set A has a proper subset B such that A is equinumerous to B then A must be infinite.

T **F** (i) Suppose that

- $P(n)$ is a sentence depending on n.
- $P(1)$ is true.
- $(\forall a, b \in \mathbf{N})\Big(\big(P(a) \text{ and } P(b)\big) \Rightarrow P(ab)\Big)$.

Then $(\forall n \in \mathbf{N})P(n)$ is true.

T **F** (j) For $\alpha \in \mathbf{R}$ let $B_\alpha = (\alpha, \infty) \times (-\infty, \alpha)$. Then

$$\bigcup_{\alpha \in \mathbf{R}} B_\alpha = \{(x, y) \in \mathbf{R} \times \mathbf{R} \,|\, x > y\}$$

2. *(10 points)* Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**

(a) Let $A = \{1, 2, 3\}$. Give two different bijections $f : A \to A$ and $g : A \to A$.

(b) Give an injection $h : \mathbf{R} \to \mathcal{P}(\mathbf{R} \times \mathbf{R})$.

(c) Give two different elements of $\mathcal{P}(\mathcal{P}(\{1, 2\}))$.

(d) Give a sequence of sets $\langle B_n \rangle_{n\in\mathbf{N}}$ such that $\bigcup_{n\in\mathbf{N}} B_n = \mathbf{Z}$ and $\bigcap_{n\in\mathbf{N}} B_n = \varnothing$.

(e) Give an element of $\{0, 1, 2\}^{\mathbf{R}}$.

(f) Give a function $f : \mathbf{N} \to \mathbf{N}$ such that $f = f^{-1}$.

(g) Give two irrational numbers $x, y \in \mathbf{R}$ such that $(x + y)$ is rational.

(h) Give an element of $\mathbf{R}^4$.

(i) For each $n \in \mathbf{N}$ let $D_n = \{k \in \mathbf{Z} \,|\, n \text{ divides } k\}$. Give an element of $\bigcap_{n\in\mathbf{N}} D_n$.

(j) Give a nonempty set $B \subset \mathbf{Q}$ such that if $x, y \in B$ then $\frac{x}{y} \in B$.

(k) Give a function $s : \mathbf{R} \to \mathbf{N}$ which is a surjection.

(l) Give a function $f : \mathbf{R} \to \mathbf{R}$ such that $f \circ f = f$ and f is not the identity function $\mathrm{id}_{\mathbf{R}}$.

3. *(5 points)* Using only the relevant definitions prove that if $\mathcal{A}$ is a nonempty set of sets and S is a set then

$$S \setminus \bigcap \mathcal{A} = \bigcup \{S \setminus A \,|\, A \in \mathcal{A}\}$$

4. *(5 points)* Using only the relevant definitions prove that for any set A the power set $\mathcal{P}(A)$ is equinumerous with the set $\{0, 1\}^A$.

5. *(5 points)* **Without refering to any theorems from the text or lecture** prove that for all $n \in \omega$

$$\sum_{k=n}^{2n} k = \frac{n(3n + 3)}{2}.$$

6. *(5 points)* Using only the relevant definitions and the following two facts prove that if x is rational and $n \in \mathbf{Z}$ then if $x^3 = n$ then x is an integer.

FACT I: If $x \in \mathbf{Q}$ then there exist $a \in \mathbf{Z}$ and $b \in \mathbf{N}$ such that $x = \frac{a}{b}$.

FACT II: If $d, n, n_1, n_2, \cdots, n_k \in \mathbf{Z}$ and $n = n_1 n_2 \cdots n_k$ and d divides n then there are $d_1, d_2, \cdots, d_k \in \mathbf{Z}$ such that d_1 divides n_1, d_2 divides n_2, $\cdots$ d_k divides n_k and $d = d_1 d_2 \cdots d_k$.

7. *(5 points)* Using **only the relevant definitions and the following fact** prove that if n and m are integers, n divides m and m divides n then $n = \pm m$.

FACT: If $x \in \mathbf{Z}$ and x divides 1 then $x \in \{-1, 1\}$.

8. *(5 points)* Let S and T are sets and define a function

$$f : \mathcal{P}(S) \times \mathcal{P}(T) \to \mathcal{P}(S \cup T)$$

by $f(A, B) = A \cup B$ for all $A \subset S$ and $B \subset T$. Show that f is a surjection.

9. *(5 points)* Given $n \in \{2, 3, 4, \cdots\}$ and $k \in \{1, 2, \cdots, n - 1\}$ let

$$A = \{1, 2, 3, \cdots, n\} \quad \text{and} \quad B = \{1, 2, \cdots, n - 1\}.$$

Let

$$M = \mathcal{P}_k(B) \cup \mathcal{P}_{k-1}(B).$$

Show that there is a surjection $f : \mathcal{P}_k(A) \to M$.