Homework 10

Reading: Chapter 6 Sections 74-81

Homework Problems:

1. Prove that the series
   \[ \sum_{n=1}^{\infty} \frac{z^n}{1 + z^{2n}} \]
   converges on both the interior and exterior of the unit circle to holomorphic functions on each region.

2. Differentiate the Maclaurin series for \(1/(1 - z)\) to give the Maclaurin series for
   \[ \frac{1}{(1 - z)^2}. \]

3. Compute the following residues:
   (a) \[ \text{Res}_{z=0} \frac{e^z - 1}{z^2}. \]
   (b) \[ \text{Res}_{z=1} \frac{z}{z^2 - 1}. \]

4. Where are the poles of \(f(z) = 1/\cos z\) and what are their orders?

5. Find all singular points of
   \[ f(z) = \frac{1}{z^3 - 3} \]
   and compute the residue of \(f\) at each of those points.

6. Recall that when \(\text{Re } z > 1\) the Riemann zeta function is given by the series
   \[ \zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} \]
   where \(1/k^z\) is taken to mean the principal value \(P.V. (1/k)^z = e^{-z \ln k}\).
   (a) Give the Taylor series at \(z = 2\) for
       \[ g_k(z) = e^{-z \ln k}. \]
   (b) Notice that for \(\text{Re } z > 1\)
       \[ \zeta(z) = \sum_{k=1}^{\infty} g_k(z). \]

       Give the Taylor series for \(\zeta(z)\) at \(z = 2\).