Math 4552
Complex Analysis

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Ohio State University

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Course Info

Course Info

- **lecturer** Nathan Broaddus
- **office** Math Tower (MW) 650
- **text** R. V. Churchill and J. W. Brown, *Complex variables and applications*, ninth edition, McGraw-Hill, New York, 2013. ISBN-13: 978-0073383170
- **room** Scott Laboratory (SO) N056
- **time** MWF 12:40pm-1:35pm
- **webpage** https://people.math.osu.edu/broaddus.9/4552/

Announcements

1. Assignment 2 due Wed. 1/28

Last time

1. Arg z vs. arg z
2. arg $zw = \arg z + \arg w$
3. computing Arg z
4. computing z^n using exponential form
5. computing $z^{1/n}$ using exponential form
6. nth roots of unity
7. the principal nth root of unity $\omega_n = e^{2\pi i/n}$

Roots of unity

Roots of unity

What are all of the mth roots of 1?
Again we will use exponential form: $1 = 1e^{i0}$
Suppose $(re^{i\theta})^m = 1e^{i0}$ then $r = \sqrt[m]{1} = 1$ and

$$m\theta = 0 + 2n\pi$$
$$\theta = \frac{2\pi n}{m}$$

List all such values of $\theta \in [0, 2\pi)$: $0, \frac{2\pi}{m}, \frac{4\pi}{m}, \cdots, \frac{(2m-2)\pi}{m}$.
and the mth roots of unity are the elements of the set:

$$\{1, e^{i2\pi/m}, e^{i4\pi/m}, \cdots, e^{i(2m-2)\pi/m}\}$$

Notice that if we set $\omega_m = e^{i2\pi/m}$ (called the **principal mth root of unity**) Then the set of all mth roots of unity is:

$$\{1, \omega_m, \omega_m^2, \cdots, \omega_m^{m-1}\}$$

Roots and roots of unity

- Suppose z_1 and z_2 are both mth roots of the complex number c.
- Then $z_1^m = z_2^m = c$ so $(\frac{z_2}{z_1})^m = 1$. Hence $\frac{z_2}{z_1}$ is an mth root of unity so there is $k \in \{0, 1, \cdots, m-1\}$ such that $\frac{z_2}{z_1} = \omega_m^k$.
- Thus $z_2 = z_1 \omega_m^k$ for some $k \in \{0, 1, \cdots, m-1\}$.
- Conversely suppose z is an mth roots of the complex number c and $k \in \{0, 1, \cdots, m-1\}$
- Then $(z\omega_m^k)^m = z^m \omega_m^{km} = c \cdot 1 = c$.
- Hence if z is an mth root of c then set of all mth roots of c is

$$\{z, z\omega_m, z\omega_m^2, \cdots, z\omega_m^{m-1}\}$$

Proposition 1 (Quadratic equation)

If $a, b, c \in \mathbf{C}$ and $b^2 - 4ac \neq 0$ and $a \neq 0$ then there are exactly two complex roots to the equation $az^2 + bz + c = 0$ given by the quadratic equation

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Proof.

- Suppose $a, b, c \in \mathbf{C}$, $b^2 - 4ac \neq 0$ and $a \neq 0$ and $az^2 + bz + c = 0$.
- Then $z^2 + \frac{b}{a} \cdot z + \frac{c}{a} = 0$
- Then $z^2 + \frac{b}{a} \cdot z + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$
- Then $z^2 + \frac{b}{a} \cdot z + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$
- Then $(z + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$
- Thus $z + \frac{b}{2a}$ is a square root of $\frac{b^2 - 4ac}{4a^2}$
- Suppose s is a square root of $b^2 - 4ac$. Then $(\frac{s}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$.

Proof of Proposition 1 *(continued)*.

- The other square root of $\frac{b^2-4ac}{4a^2}$ must be $\frac{s}{2a}\omega_2 = \frac{s}{2a}e^{i\pi} = -\frac{s}{2a}$.
- Thus if z satisfies $az^2 + bz + c = 0$ then either $z + \frac{b}{2a} = \frac{s}{2a}$ or $z + \frac{b}{2a} = -\frac{s}{2a}$.
- Hence if $z = \frac{-b+s}{2a}$ or $z = \frac{-b-s}{2a}$.
- That is $z = \frac{-b+\sqrt{b^2-4ac}}{2a}$ or $z = \frac{-b-\sqrt{b^2-4ac}}{2a}$.
- Conversely if $z = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ then z satisfies $az^2 + bz + c = 0$ by direct computation (do this yourself).

□

Regions in the complex plane

Definition 2 (ε-neighborhood)

For $\varepsilon > 0$ an ε-**neighborhood** of the point z_0 is the set

$$B_\varepsilon(z_0) = \{z \in \mathbf{C} \mid |z - z_0| < \varepsilon\}.$$

The **deleted** ε-**neighborhood** of the point z_0 is the set

$$B_\varepsilon(z_0) - \{z_0\} = \{z \in \mathbf{C} \mid 0 < |z - z_0| < \varepsilon\}.$$

Definition 3

Let $S \subset \mathbf{C}$.

1. z_0 is an **interior point** of S if there is $\varepsilon > 0$ such that $B_\varepsilon(z_0) \subset S$.
2. z_0 is an **exterior point** of S if there is $\varepsilon > 0$ such that $B_\varepsilon(z_0) \cap S = \varnothing$.
3. z_0 is an **boundary point** of S if it is neither an interior nor exterior point of S.

Definition 4 (Open and closed sets)

1. $S \subset \mathbf{C}$ is **open** if it does not contain any of its boundary points.
2. $S \subset \mathbf{C}$ is **closed** if it contains all of its boundary points.

Definition 5

1. The **boundary** of a set $S \subset \mathbf{C}$ is the set

$$\text{bnd}\, S = \{z \in \mathbf{C} | \; z \text{ is a boundary point of } S\}$$

2. The **interior** of a set $S \subset \mathbf{C}$ is the set

$$\text{int}\, S = S - \text{bnd}\, S$$

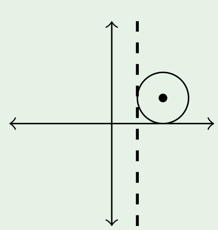
3. The **closure** of a set $S \subset \mathbf{C}$ is the set

$$\text{cls}\, S = S \cup \text{bnd}\, S$$

Example 6 (Exterior point)

Show that $2 + i$ is an exterior point of the set $S = \{z \in \mathbf{C} | \; \text{Re}\, z < 1\}$

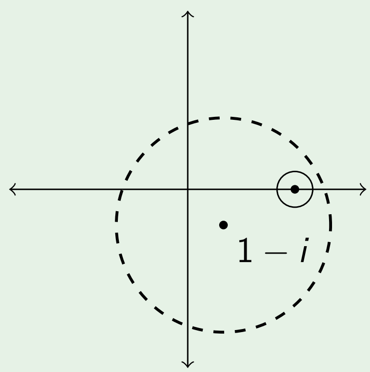
- We must find a neighborhood of $2 + i$ which is disjoint from S.
- Suppose $z \in B_1(2 + i)$.
- Then $\text{Re}(2 + i - z) < |2 + i - z| < 1$.
- Thus $2 - \text{Re}(z) < 1$.
- Thus $-2 + \text{Re}(z) > -1$.
- Thus $\text{Re}(z) > 1$.
- Hence $z \notin S$.
- Therefore $S \cap B_1(2 + i) = \varnothing$.
- Hence $2 + i$ is an exterior point of S.



Example 7 (Interior point)

Show that 3 is an interior point of the set $B_3(1-i)$

- We must find a neighborhood of 3 which is contained in $B_3(1-i)$.
- Suppose $z \in B_{\frac{1}{2}}(3)$.
- Then $|1-i-z| = |1-i-3+3-z| \leq |1-i-3| + |3-z| = |-2-i| + |3-z| < \sqrt{5} + \frac{1}{2} < 3$.
- Hence $z \in B_3(1-i)$.
- Therefore $B_{\frac{1}{5}}(3) \subset B_3(1-i)$.
- Hence $2+i$ is an exterior point of S.



Example 8 (Open set)

Show that the set $B_3(1-i)$ is open.

- Suppose $z \in B_3(1-i)$.
- Then $|1-i-z| < 3$.
- Let $\varepsilon = 3 - |1-i-z|$
- Then if $w \in B_\varepsilon(z)$ we have $|1-i-w| = |1-i-z+z-w| \leq |1-i-z| + |z-w| < |1-i-z| + 3 - |1-i-z| = 3$.
- Therefore $B_\varepsilon(z) \subset B_3(1-i)$.
- Hence z is an interior point of $B_3(1-i)$ and not a boundary point.