Math 4552
Complex Analysis

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Course Info

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Announcements

1. Assignment 2 due Wed. 1/28
Last time

1. Arg \(z\) vs. \(\arg z\)
2. \(\arg zw = \arg z + \arg w\)
3. computing \(\text{Arg } z\)
4. computing \(z^n\) using exponential form
5. computing \(z^{1/n}\) using exponential form
6. \(n\)th roots of unity
7. the principal \(n\)th root of unity \(\omega_n = e^{2\pi i/n}\)

Roots of unity

What are all of the \(m\)th roots of 1?
Again we will use exponential form: \(1 = 1e^{i0}\)
Suppose \((re^{i0})^m = 1e^{i0}\) then \(r = \sqrt[m]{1} = 1\) and

\[ m\theta = 0 + 2n\pi \]
\[ \theta = \frac{2\pi n}{m} \]

List all such values of \(\theta \in [0, 2\pi)\): \(0, \frac{2\pi}{m}, \frac{4\pi}{m}, \ldots, \frac{(2m-2)\pi}{m}\).
and the \(m\)th roots of unity are the elements of the set:

\[ \{1, e^{i2\pi/m}, e^{i4\pi/m}, \ldots, e^{i(2m-2)\pi/m}\} \]

Notice that if we set \(\omega_m = e^{i2\pi/m}\) (called the principal \(m\)th root of unity) Then the set of all \(m\)th roots of unity is:

\[ \{1, \omega_m, \omega_m^2, \ldots, \omega_m^{m-1}\} \]
Roots and roots of unity

- Suppose $z_1$ and $z_2$ are both $m$th roots of the complex number $c$.
- Then $z_1^m = z_2^m = c$ so $(z_2 / z_1)^m = 1$. Hence $z_2 / z_1$ is an $m$th root of unity so there is $k \in \{0, 1, \cdots, m - 1\}$ such that $z_2 / z_1 = \omega^k_m$.
- Thus $z_2 = z_1 \omega^k_m$ for some $k \in \{0, 1, \cdots, m - 1\}$.
- Conversely suppose $z$ is an $m$th roots of the complex number $c$ and $k \in \{0, 1, \cdots, m - 1\}$
  - Then $(z \omega^k_m)^m = z^m \omega^{km}_m = c \cdot 1 = c$.
  - Hence if $z$ is an $m$th root of $c$ then set of all $m$th roots of $c$ is
  \[
  \{z, z \omega_m, z \omega^2_m, \cdots, z \omega^{m-1}_m\}
  \]

Proposition 1 (Quadratic equation)

If $a, b, c \in \mathbb{C}$ and $b^2 - 4ac \neq 0$ and $a \neq 0$ then there are exactly two complex roots to the equation $az^2 + bz + c = 0$ given by the quadratic equation

\[
z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}
\]

Proof.

- Suppose $a, b, c \in \mathbb{C}$, $b^2 - 4ac \neq 0$ and $a \neq 0$ and $az^2 + bz + c = 0$.
- Then $z^2 + \frac{b}{a} \cdot z + \frac{c}{a} = 0$
- Then $z^2 + \frac{b}{a} \cdot z + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$
- Then $z^2 + \frac{b}{a} \cdot z + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$
- Then $(z + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$
- Thus $z + \frac{b}{2a}$ is a square root of $\frac{b^2 - 4ac}{4a^2}$
- Suppose $s$ is a square root of $b^2 - 4ac$. Then $(\frac{s}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$. 
Proof of Proposition 1 (continued).

- The other square root of \( \frac{b^2 - 4ac}{4a^2} \) must be \( \frac{s}{2a} \omega_2 = \frac{s}{2a} e^{i\pi} = -\frac{s}{2a} \).
- Thus if \( z \) satisfies \( az^2 + bz + c = 0 \) then either \( z + \frac{b}{2a} = \frac{s}{2a} \) or \( z + \frac{b}{2a} = -\frac{s}{2a} \).
- Hence if \( z = \frac{-b + s}{2a} \) or \( z = \frac{-b - s}{2a} \).
- That is \( z = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \) or \( z = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \).
- Conversely if \( z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) then \( z \) satisfies \( az^2 + bz + c = 0 \) by direct computation (do this yourself).

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**Regions in the complex plane**

**Definition 2 (\( \varepsilon \)-neighborhood)**

For \( \varepsilon > 0 \) an \( \varepsilon \)-neighborhood of the point \( z_0 \) is the set

\[
B_\varepsilon(z_0) = \{ z \in \mathbb{C} | |z - z_0| < \varepsilon \}.
\]

The deleted \( \varepsilon \)-neighborhood of the point \( z_0 \) is the set

\[
B_\varepsilon(z_0) - \{ z_0 \} = \{ z \in \mathbb{C} | 0 < |z - z_0| < \varepsilon \}.
\]

**Definition 3**

Let \( S \subset \mathbb{C} \).

1. \( z_0 \) is an **interior point** of \( S \) if there is \( \varepsilon > 0 \) such that \( B_\varepsilon(z_0) \subset S \).
2. \( z_0 \) is an **exterior point** of \( S \) if there is \( \varepsilon > 0 \) such that \( B_\varepsilon(z_0) \cap S = \emptyset \).
3. \( z_0 \) is an **boundary point** of \( S \) if it is neither an interior nor exterior point of \( S \).
**Definition 4 (Open and closed sets)**

1. **Closed set**: A set $S \subset \mathbb{C}$ is **open** if it does not contain any of its boundary points.
2. **Closed set**: A set $S \subset \mathbb{C}$ is **closed** if it contains all of its boundary points.

**Definition 5**

1. The **boundary** of a set $S \subset \mathbb{C}$ is the set
   \[ \text{bnd } S = \{ z \in \mathbb{C} \mid z \text{ is a boundary point of } S \} \]
2. The **interior** of a set $S \subset \mathbb{C}$ is the set
   \[ \text{int } S = S - \text{bnd } S \]
3. The **closure** of a set $S \subset \mathbb{C}$ is the set
   \[ \text{cls } S = S \cup \text{bnd } S \]

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**Example 6 (Exterior point)**

Show that $2 + i$ is an exterior point of the set $S = \{ z \in \mathbb{C} \mid \text{Re } z < 1 \}$

- We must find a neighborhood of $2 + i$ which is disjoint from $S$.
- Suppose $z \in B_1(2 + i)$.
- Then $\text{Re}(2 + i - z) < |2 + i - z| < 1$.
- Thus $2 - \text{Re}(z) < 1$.
- Thus $-2 + \text{Re}(z) > -1$.
- Thus $\text{Re}(z) > 1$.
- Hence $z \notin S$.
- Therefore $S \cap B_1(2 + i) = \emptyset$.
- Hence $2 + i$ is an exterior point of $S$. 
Example 7 (Interior point)

Show that 3 is an interior point of the set $B_3(1 - i)$

- We must find a neighborhood of 3 which is contained in $B_3(1 - i)$.
- Suppose $z \in B_{\frac{1}{2}}(3)$.
- Then $|1 - i - z| = |1 - i - 3 + 3 - z| \leq |1 - i - 3| + |3 - z| = |-2 - i| + |3 - z| < \sqrt{5} + \frac{1}{2} < 3$.
- Hence $z \in B_3(1 - i)$.
- Therefore $B_{\frac{1}{2}}(3) \subset B_3(1 - i)$.
- Hence $2 + i$ is an exterior point of $S$.

Example 8 (Open set)

Show that the set $B_3(1 - i)$ is open.

- Suppose $z \in B_3(1 - i)$.
- Then $|1 - i - z| < 3$.
- Let $\varepsilon = 3 - |1 - i - z|$
- Then if $w \in B_\varepsilon(z)$ we have $|1 - i - w| = |1 - i - z + z - w| \leq |1 - i - z| + |z - w| < |1 - i - z| + 3 - |1 - i - z| = 3$.
- Therefore $B_\varepsilon(z) \subset B_3(1 - i)$.
- Hence $z$ is an interior point of $B_3(1 - i)$ and not a boundary point.