1 Integration

1.1 Functions from R to C

Let \( w(t) = u(t) + iv(t) \) be a complex-valued function of a real variable. We say that the function is continuous, differentiable, integrable, etc. if both \( u \) and \( v \) are. Its derivative (if it exists) is \( w'(t) = u'(t) + iv'(t) \) and its definite integral (if it exists) is

\[
\int_a^b w(t) \, dt = \int_a^b u(t) \, dt + i \int_a^b v(t) \, dt
\]

Directly from multivariable calculus we have a fundamental theorem of calculus

**Proposition 1.1.** If \( W(t) \) is a differentiable path with \( W'(t) = w(t) \) then

\[
\int_a^b w(t) \, dt = W(b) - W(a)
\]

**Example 1.2.** Compute \( w'(t) \) if \( w(t) = e^t - t + i\sin(t) \):

\[
w'(t) = \frac{d}{dt}(e^t - t) + i \frac{d}{dt} \sin t
= e^t - 1 + i\cos t
\]

**Example 1.3.** Compute \( w'(t) \) if \( w(t) = \sin(t - it) \):

\[
w(t) = \sin(t - it)
= \sin t \cosh(-t) + i \cos t \sinh(-t)
= \sin t \cosh t - i \cos t \sinh t
\]

\[
w'(t) = \frac{d}{dt} \sin t \cosh t - i \frac{d}{dt} \cos t \sinh t
= \cos t \cosh t + \sin t \sinh t - i(-\sin t \sinh t + \cos t \cosh t)
= \frac{\cos t \cosh t + \sin t \sinh t + i(\sin t \sinh t - \cos t \cosh t)}{\sqrt{1 + i^2}}
\]

This was a lot of work when we’d like to just use the chain rule to get:

\[
w'(t) = (1 - i) \cos(t - it).
\]

The next theorem says this is ok
Theorem 1.4. If \( w(t) \) is differentiable and \( f(z) \) is holomorphic then

\[
\frac{df}{dt} = w'(t)f(w(t))
\]

Proof.

- Suppose \( f(z) = u(x, y) + iv(x, y) \) is holomorphic and \( w(t) = x(t) + iy(t) \) is differentiable.
- Then \( f(w(t)) = u(x(t), y(t)) + iv(x(t), y(t)) \)

\[
\frac{df}{dt} = \frac{d}{dt} \left( u(x(t), y(t)) + iv(x(t), y(t)) \right)
= \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + i \frac{\partial v}{\partial x} \frac{dx}{dt} + i \frac{\partial v}{\partial y} \frac{dy}{dt}
= \left( \frac{dx}{dt} + i \frac{dy}{dt} \right) \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)
= w'(t)f(w(t))
\]

Example 1.5. Compute \( w'(t) \) if \( w(t) = e^{i/t} \sin(t + it^2) \):

\[
w'(t) = \left( -\frac{ie^{i/t} \sin(t + it^2)}{t^2} \right) + e^{i/t} (1 + 2it) \cos(t + it^2)
\]

Example 1.6. Compute \( \int_0^{\pi/2} e^{i\theta} \, d\theta \)

\[
\int_0^{\pi/2} e^{i\theta} \, d\theta = \int_0^{\pi/2} \cos \theta + i \sin \theta \, d\theta
= \int_0^{\pi/2} \cos \theta \, d\theta + i \int_0^{\pi/2} \sin \theta \, d\theta
= \left. \sin \theta \right|_0^{\pi/2} - i \left. \cos \theta \right|_0^{\pi/2}
= \sin \frac{\pi}{2} - \sin 0 - i(\cos \frac{\pi}{2} - \cos 0)
= 1 - 0 - i(0 - 1)
= 1 + i
\]

Alternatively, if \( W(\theta) = e^{i\theta} \), then \( W'(\theta) = e^{i\theta} \cdot i = e^{i\theta} \). Thus

\[
\int_0^{\pi/2} e^{i\theta} \, d\theta = e^{i\theta} \bigg|_0^{\pi/2} = e^{\pi/2} i - e^{i\theta} \bigg|_0^{\pi/2} = i - \frac{1}{i} = 1 + i
\]