Complex Analysis – Lecture 14

Prof. Broaddus

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1 Integration

1.1 Functions from R to C

Let

$$w(t) = u(t) + iv(t)$$

be a complex-valued function of a real variable. We say that the function is **continuous**, **differentiable**, **integrable**, etc. if both u and v are. Its **derivative** (if it exists) is

$$w'(t) = u'(t) + iv'(t)$$

and its **definite integral** (if it exists) is

$$\int_a^b w(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t$$

Directly from multivariable calculus we have a **fundamental theorem of calculus**

Proposition 1.1. *If $W(t)$ is a differentiable path with $W'(t) = w(t)$ then*

$$\int_a^b w(t)\,\mathrm{d}t = W(b) - W(a)$$

Example 1.2. Compute $w'(t)$ if $w(t) = e^t - t + i(\sin t)$:

$$\begin{aligned} w'(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t}(e^t - t) + i\tfrac{\mathrm{d}}{\mathrm{d}t}\sin t \\ &= \boxed{e^t - 1 + i\cos t} \end{aligned}$$

Example 1.3. Compute $w'(t)$ if $w(t) = \sin(t - it)$:

$$\begin{aligned} w(t) &= \sin(t - it) \\ &= \sin t\cosh(-t) + i\cos t\sinh(-t) \\ &= \sin t\cosh t - i\cos t\sinh t \\ w'(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\sin t\cosh t - i\tfrac{\mathrm{d}}{\mathrm{d}t}\cos t\sinh t \\ &= \cos t\cosh t + \sin t\sinh t - i(-\sin t\sinh t + \cos t\cosh t) \\ &= \boxed{\cos t\cosh t + \sin t\sinh t + i(\sin t\sinh t - \cos t\cosh t)} \end{aligned}$$

This was a lot of work when we'd like to just use the chain rule to get:

$$w'(t) = (1 - i)\cos(t - it).$$

The next theorem says this is ok

Theorem 1.4. *If $w(t)$ is differentiable and $f(z)$ is holomorphic then*

$$\tfrac{\mathrm{d}}{\mathrm{d}t} f(w(t)) = w'(t) f'(w(t))$$

Proof.

- Suppose $f(z) = u(x,y) + iv(x,y)$ is holomorphic and $w(t) = x(t) + iy(t)$ is differentiable.
- Then $f(w(t)) = u(x(t), y(t)) + iv(x(t), y(t))$

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} f(w(t)) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\Big(u(x(t), y(t)) + iv(x(t), y(t))\Big) \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}(u(x(t), y(t)) + i\tfrac{\mathrm{d}}{\mathrm{d}t} v(x(t), y(t)) \\
&= \tfrac{\partial u}{\partial x}\tfrac{\mathrm{d}x}{\mathrm{d}t} + \tfrac{\partial u}{\partial y}\tfrac{\mathrm{d}y}{\mathrm{d}t} + i\tfrac{\partial v}{\partial x}\tfrac{\mathrm{d}x}{\mathrm{d}t} + i\tfrac{\partial v}{\partial y}\tfrac{\mathrm{d}y}{\mathrm{d}t} \\
&= \tfrac{\partial u}{\partial x}\tfrac{\mathrm{d}x}{\mathrm{d}t} - \tfrac{\partial v}{\partial x}\tfrac{\mathrm{d}y}{\mathrm{d}t} + i\tfrac{\partial v}{\partial x}\tfrac{\mathrm{d}x}{\mathrm{d}t} + i\tfrac{\partial u}{\partial x}\tfrac{\mathrm{d}y}{\mathrm{d}t} \\
&= (\tfrac{\mathrm{d}x}{\mathrm{d}t} + i\tfrac{\mathrm{d}y}{\mathrm{d}t})(\tfrac{\partial u}{\partial x} + i\tfrac{\partial v}{\partial x}) \\
&= w'(t) f(w(t))
\end{aligned}$$

□

Example 1.5. Compute $w'(t)$ if $w(t) = e^{i/t}\sin(t + it^2)$:

$$w'(t) = \boxed{\tfrac{-ie^{i/t}\sin(t+it^2)}{t^2} + e^{i/t}(1+2it)\cos(t+it^2)}$$

Example 1.6. Compute $\int_0^{\pi/2} e^{i\theta}\,\mathrm{d}\theta$

$$\begin{aligned}
\int_0^{\pi/2} e^{i\theta}\,\mathrm{d}\theta &= \int_0^{\pi/2} \cos\theta + i\sin\theta\,\mathrm{d}\theta \\
&= \int_0^{\pi/2} \cos\theta\,\mathrm{d}\theta + i\int_0^{\pi/2} \sin\theta\,\mathrm{d}\theta \\
&= \sin\theta\Big|_0^{\pi/2} - i\cos\theta\Big|_0^{\pi/2} \\
&= \sin\tfrac{\pi}{2} - \sin 0 - i(\cos\tfrac{\pi}{2} - \cos 0) \\
&= 1 - 0 - i(0-1) \\
&= \boxed{1+i}
\end{aligned}$$

Alternatively, if $W(\theta) = \frac{e^{i\theta}}{i}$ then $W'(\theta) = \frac{e^{i\theta}}{i}\cdot i = e^{i\theta}$. Thus

$$\int_0^{\pi/2} e^{i\theta}\,\mathrm{d}\theta = \frac{e^{i\theta}}{i}\Big|_0^{\frac{\pi}{2}} = \frac{e^{i\pi/2}}{i} - \frac{e^{i0}}{i} = \frac{i}{i} - \frac{1}{i} = 1+i$$