Complex Analysis – Lecture 15

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1 Integration

1.1 Integrals and contour integrals

In Lecture 14 we proved the following theorem:

Theorem 1.1. *If $w(t)$ is differentiable and $f(z)$ is holomorphic then*

$$\tfrac{\mathrm{d}}{\mathrm{d}t} f(w(t)) = w'(t) f'(w(t))$$

This is the key to differentiating and integrating complex-valued functions of a real variable using the same techniques (substitution, integration by parts) as we would a real-valued function of a real variable.

Example 1.2. Compute $w'(t)$ if $w(t) = e^{i/t}\sin(t + it^2)$:

$$w'(t) = \boxed{\tfrac{-ie^{i/t}\sin(t+it^2)}{t^2} + e^{i/t}(1+2it)\cos(t+it^2)}$$

Example 1.3. Compute $\int_0^{\pi/2} e^{i\theta}\,\mathrm{d}\theta$

- Using the definition of the definite integral we get:

$$\begin{aligned}
\int_0^{\pi/2} e^{i\theta}\,\mathrm{d}\theta &= \int_0^{\pi/2} \cos\theta + i\sin\theta\,\mathrm{d}\theta \\
&= \int_0^{\pi/2} \cos\theta\,\mathrm{d}\theta + i\int_0^{\pi/2} \sin\theta\,\mathrm{d}\theta \\
&= \sin\theta\Big|_0^{\pi/2} - i\cos\theta\Big|_0^{\pi/2} \\
&= \sin\tfrac{\pi}{2} - \sin 0 - i(\cos\tfrac{\pi}{2} - \cos 0) \\
&= 1 - 0 - i(0-1) \\
&= \boxed{1+i}
\end{aligned}$$

- Alternatively, if $W(\theta) = \frac{e^{i\theta}}{i}$ then applying Theorem 1.1 we get $W'(\theta) = \frac{e^{i\theta}}{i}\cdot i = e^{i\theta}$. Thus

$$\int_0^{\pi/2} e^{i\theta}\,\mathrm{d}\theta = \frac{e^{i\theta}}{i}\Big|_0^{\frac{\pi}{2}} = \frac{e^{i\pi/2}}{i} - \frac{e^{i0}}{i} = \frac{i}{i} - \frac{1}{i} = \boxed{1+i}$$

1.2 Arcs in C

Definition 1.4 (Properties of arcs)**.**

1. An **arc** (or **contour**) in **C** is the oriented (directed) image of a continuous function $\alpha : [a, b] \to \mathbf{C}$.
2. A **parametrization** of an arc is a particular function $\alpha : [a, b] \to \mathbf{C}$ whose image is the arc in question.
3. A **simple arc** is an arc which does not cross itself ($\alpha(x) = \alpha(y)$ iff $x = y$)
4. A **simple closed curve** in **C** an arc $\alpha : [a, b] \to \mathbf{C}$ with $\alpha(a) = \alpha(b)$ which does not cross itself otherwise.
5. An simple closed curve is **positively oriented** if it proceeds in the counter clockwise direction. Otherwise it is **negatively oriented**.

A single arc can be parametrized in infinitely many ways

Example 1.5. Give 3 different parametrizations of the straight line segment starting at $6 - i$ and ending at $1 - 3i$.

1. $\boxed{\alpha(t) = (1 - t)(6 - i) + t(1 - 3i) = (-5 - 2i)t + 6 - i \text{ where } t \in [0, 1]}$
2. $\boxed{\beta(t) = (1 - \frac{t}{10}))(6 - i) + \frac{t}{10}(1 - 3i) \text{ where } t \in [0, 10]}$
3. $\boxed{\gamma(t) = (1 - \sin(t))(6 - i) + \sin t(1 - 3i) \text{ where } t \in [0, \frac{\pi}{2}]}$

Example 1.6. Give 2 different parametrizations of upper half-circle of radius 2 centered at 0 starting at $z = 2$ and ending at $z = -2$.

1. $\boxed{\rho(t) = 2e^{it} \text{ where } t \in [0, \pi]}$
2. $\boxed{\eta(t) = 2e^{i2\pi t} \text{ where } t \in [0, \frac{1}{2}]}$

Definition 1.7 (Length of an arc)**.** The length of the arc $z(t)$ for $t \in [a, b]$ is

$$L = \int_a^b |z'(t)| \, \mathrm{d}t$$

Example 1.8. Find the length of the half-circle of radius 2 centered at $z = 0$ starting at $z = 1$ and ending at $z = -1$.

- $z(t) = 2e^{it}$ on $[0, \pi]$ so $z'(t) = 2ie^{it}$ and

$$\begin{aligned} L &= \textstyle\int_0^\pi |2ie^{it}| \, \mathrm{d}t \\ &= \textstyle\int_0^\pi 2 \, \mathrm{d}t \\ &= 2t|_0^\pi \\ &= \boxed{2\pi} \end{aligned}$$

1.3 Contour integrals

Given a contour C it would be nice to define integral of $f(z)$ over C in a way that does not depend on the parametrization of C.

Definition 1.9 (Countour integral)**.** If $f(z)$ is a function and C is a contour with parametrization $\gamma(t)$ then the **contour integral** of f over C is:

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

Example 1.10. Compute the contour integral

$$\int_C \frac{z}{-5-2i}\,\mathrm{d}z$$

where C is the straight line segment starting at $6-i$ and ending at $1-3i$ using the parametrization $z(t) = (-5-2i)t + 6 - i$ where $t \in [0,1]$

- $z'(t) = -5-2i$ so

$$\begin{aligned}
\int_C \frac{z}{-5-2i}\,\mathrm{d}z &= \textstyle\int_0^1 \frac{z(t)}{-5-2i} \cdot z'(t)\,\mathrm{d}t \\
&= \textstyle\int_0^1 \frac{(-5-2i)t+6-i}{-5-2i} \cdot (-5-2i)\,\mathrm{d}t \\
&= \textstyle\int_0^1 (-5-2i)t + 6 - i\,\mathrm{d}t \\
&= \left.\textstyle\frac{(-5-2i)t^2}{2} + (6-i)t\right|_0^1 \\
&= \boxed{\textstyle\frac{-5-2i}{2} + (6-i)}
\end{aligned}$$

Example 1.11. Compute the same contour integral

$$\int_C \frac{z}{-5-2i}\,\mathrm{d}z$$

where C is the straight line segment starting at $6-i$ and ending at $1-3i$ using the parametrization $z(\theta) = (-5-2i)\sin\theta + 6 - i$ where $\theta \in [0, \frac{\pi}{2}]$

- $z'(\theta) = (-5-2i)\cos\theta$ so

$$\begin{aligned}
\int_C \frac{z}{-5-2i}\,\mathrm{d}z &= \textstyle\int_0^{\frac{\pi}{2}} \frac{z(t)}{-5-2i} \cdot z'(t)\,\mathrm{d}t \\
&= \textstyle\int_0^{\frac{\pi}{2}} \frac{(-5-2i)\sin\theta+6-i}{-5-2i} \cdot (-5-2i)\cos\theta\,\mathrm{d}\theta \\
&= \textstyle\int_0^{\frac{\pi}{2}} ((-5-2i)\sin\theta + 6 - i) \cdot \cos\theta\,\mathrm{d}\theta \\
&= \textstyle\int_0^1 (-5-2i)t + 6 - i\,\mathrm{d}t \\
&= \left.\textstyle\frac{(-5-2i)t^2}{2} + (6-i)t\right|_0^1 \\
&= \boxed{\textstyle\frac{-5-2i}{2} + (6-i)}
\end{aligned}$$

1.4 Bounds on integrals

One of most important mathematical skills is bounding values. For example the triangle inequality bounds the modulus of the sum of two numbers based on the modulus of the numbers

$$|z+w| \leq |z| + |w|$$

This is called a **tight bound** because without knowing more about z and w the sum $|z| + |w|$ this is the lowest possible bound we can give for $|z+w|$. Here is another tight bound

Theorem 1.12. *If $w(t)$ is integrable then*

$$\left|\int_a^b w(t)\,\mathrm{d}t\right| \leq \int_a^b |w(t)|\,\mathrm{d}t$$

Theorem 1.13. *If C is a contour and $|f(z)| \leq M$ for all $z \in C$ then*

$$\left|\int_C f(z)\,\mathrm{d}z\right| \leq ML$$

where L is the length of the contour C.

Example 1.14. Bound the integral

$$\int_1^3 \frac{t^3 + it^2 + 1}{t+2}\,\mathrm{d}t$$

- If $|t| \leq 3$ then from midterm $|t^3 + it^2 + 1| \leq 1 \cdot \frac{1-3^4}{1-3} = 40$
- If $1 < t < 3$ then $3 < t + 2 < 5$ so $|t + 2| \geq 3$. Hence $\frac{1}{|t+2|} \leq \frac{1}{3}$
- Thus $\left|\frac{t^3+it^2+1}{t+2}\right| = |t^3 + it^2 + 1| \cdot \frac{1}{|t+2|} \leq 40 \cdot \frac{1}{3} = \frac{40}{3}$
- $$\left|\int_1^3 \frac{t^3+it^2+1}{t+2}\,\mathrm{d}t\right| \leq \int_1^3 \left|\frac{t^3+it^2+1}{t+2}\right|\,\mathrm{d}t \leq \int_1^3 \frac{40}{3}\,\mathrm{d}t = 8t|_1^3 = \boxed{\frac{80}{3}}$$

Example 1.15. Bound the contour integral

$$\int_C \frac{z^3 + iz^2 + 1}{z+2}\,\mathrm{d}z$$

where C is the half circle of radius 1 centered at 2 from 1 to 3.

- If $|z| \leq 3$ then from midterm $|z^3 + iz^2 + 1| \leq 1 \cdot \frac{1-3^4}{1-3} = 40$
- If $1 < \operatorname{Re} z < 3$ then $3 < \operatorname{Re} z + 2 < 5$ so $3 < \operatorname{Re}(z + 2) \leq |z + 2|$
- Hence $\frac{1}{|t+2|} \leq \frac{1}{3}$
- Thus $\left|\frac{z^3+iz^2+1}{z+2}\right| = |z^3 + iz^2 + 1| \cdot \frac{1}{|z+2|} \leq 40 \cdot \frac{1}{3} = \frac{40}{3}$
- Length of C is π (half circumference of circle with radius 1)
- $$\left|\int_C \frac{z^3 + iz^2 + 1}{z+2}\,\mathrm{d}z\right| \leq \boxed{\pi \cdot \frac{40}{3}}$$