Math 5801
General Topology and Knot Theory

Nathan Broaddus

Ohio State University

August 22, 2012

Course Info

Textbook (required)

J. R. Munkres, *Topology (2nd Edition)*, Prentice Hall, Englewood Cliffs, NJ, 2000.

ISBN-10: 0131816292

Reading for Friday

Chapter 1.1-1.3, pgs. 3-28

HW 1 for Monday

- Chapter 1.1: 3a, 4c
- Chapter 1.2: 2a-b, 4a-b, 5a
- Chapter 1.3: 2, 4, 11

Logical Statements

A *logical statement* (or simply a *statement*) is a mathematical assertion that is either **true** or **false**.

Examples 1

1. $1 + 2 = 3$. **true**
2. $e^7 = \pi$. **false**
3. $x = 2$ implies $x > 1$. **true**
4. (3 is even) and (6 is even). **false**
5. (3 is even) or (6 is even). **true**
6. It is not true that $1 + 1 = 2$. **false**
7. $x = y$. **?**

In 3-6 above we used *logical operators.*

Logical operators

If P and Q are statements then:

1. (P and Q) is true only if both P is true and Q is true.
2. (P or Q) is true unless both P is false and Q is false.
3. (not P) is true if P is false.
4. (P implies Q) is true unless P is true and Q is false.
5. (P iff Q) is true only if truth values of P and Q agree.

Notation

1. $P \wedge Q$ means "P and Q"
2. $P \vee Q$ means "P or Q"
3. $\neg P$ means "it is not true that P"
4. $P \implies Q$ means "P implies Q"
5. $P \iff Q$ means "P if and only if Q" means "P iff Q"

Logical Equivalence

For any statements P and Q the following statements are true:

1. $\neg(P \wedge Q) \iff (\neg P \vee \neg Q)$
2. $\neg(P \vee Q) \iff (\neg P \wedge \neg Q)$
3. $(P \implies Q) \iff (\neg P \vee Q)$
4. $(P \iff Q) \iff ((P \wedge Q) \vee (\neg P \wedge \neg Q))$
5. $(P \implies Q) \iff (\neg Q \implies \neg P)$ (proof by contraposition)
6. $(P \implies (Q \wedge \neg Q)) \iff \neg P$ (proof by contradiction)

- *Contrapositive* of $P \implies Q$ is:

$$\neg Q \implies \neg P$$

- *Converse* of $P \implies Q$ is:

$$Q \implies P$$

Quantifiers

More complex statements can be constructed using quantifiers:

$\forall$ "for all" or "for every"

and

$\exists$ "there exists" or "there is some"

Examples 2

1. $\forall x \in \mathbf{R}, x^2 \geq 0$.
2. $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}$ s.t. $x + y = 0$.
3. $\forall n \in \mathbf{Z}, \exists y \in \mathbf{R}$ s.t. $y^2 = n$.
4. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in \mathbf{R}\ (|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon)$.

Quantifiers

For any set A and statement $P(x)$ which depends on x we can construct new statements:

$$\forall x \in A, P(x) \qquad \text{"for all } x \in A, P(x) \text{ is true"}$$

and

$$\exists x \in A, P(x) \qquad \text{"There is at least one } x \in A \text{ such that } P(x) \text{ is true"}$$

$P(x)$ may itself be a quantified statement:

Examples 3

1. $x = y$ **?**
2. $\exists x \in \mathbf{R}, x = y$ **?**
3. $\forall y \in \mathbf{R}, \exists x \in \mathbf{R}$ s.t. $x = y$ **true**

Bound vs. Unbound Variables

- In the statement $\forall x \in A, P(x)$. The *scope of* x is the statement $P(x)$.
- Within its scope x has meaning and is said to be a *bound variable*.
- Otherwise x is an *unbound variable*.
- It's forbidden to quantify over a bound variable.

Example 4 (Scopes of variables)

$$\forall x \in \mathbf{R}, \underbrace{\exists y \in \mathbf{R} \text{ s.t. } \overbrace{x + y = 0}^{\text{scope of } y}}_{\text{scope of } x}.$$

Bound vs. Unbound Variables

Common source of nonsense is problems with variable scope.

Bound variables

1. $\forall x \in \mathbf{R}, \exists x \in \mathbf{R}$ s.t. $x = x$. **nonsense**
2. $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}$ s.t. $x = x$. **true**
3. $(\forall x \in \mathbf{R}, x = x)$ and $(x = 5)$. **?**
4. $\forall x \in \mathbf{R}, (x = x$ and $x = 5)$. **false**

Bound vs. Unbound Variables

In proofs we bind variables using words like "suppose" and "let" as well as "for all" and "there exists."

Example 5 (Proof that $\sqrt{2}$ is irrational)

Suppose $x \in \mathbf{Q}$ and $x^2 = 2$. Then there exist $p, q \in \mathbf{Z}$ with $q \neq 0$ and p and q relatively prime s.t. $x = \frac{p}{q}$. Then $\left(\frac{p}{q}\right)^2 = 2$ so $p^2 = 2q^2$. Hence p^2 is even. It follows that p is even so $p = 2k$ for some $k \in \mathbf{Z}$. Therefore $(2k)^2 = 2q^2$ so $2k^2 = q^2$. Thus q^2 is even which implies q is even. Thus 2 divides p and q. It follows that p and q are not relatively prime. But p and q are relatively prime. CONTRADICTION.

Thus it cannot be simultaneously true that $x \in \mathbf{Q}$ and $x^2 = 2$. $\square$

Negating Quantified Statements

Negation of quantifiers

1. $\neg(\exists x \in A, \text{ s.t. } P(x))$ is logically equivalent to $\forall x \in A, \neg P(x)$.
2. $\neg(\forall x \in A, P(x))$ is logically equivalent to $\exists x \in A, \text{ s.t. } \neg P(x)$.

- Fix a universe $\mathfrak{U}$.
- Everything in $\mathfrak{U}$ is a set.
- Any element of a set is itself a set.
- For example $\varnothing \in \{\varnothing, \{\{\varnothing\}\}\}$
- Any quantified statement (like $\exists x, x \in x$) is quantified over our entire universe $\mathfrak{U}$.
- We will assume $\mathfrak{U}$ satisfies the following collection of axioms (called ZFC)

Zermelo-Fraenkel Axioms

(EXT) *(Extensionality Axiom)* Sets are determined by their elements:

For all sets A and B, $(A = B)$ iff $(\forall x,\ x \in A$ iff $x \in B)$

(PAIR) *(Pair Set Axiom)* The set $\{x, y\}$ exists:

For all x and y, there is A s.t. $\forall z, [z \in A$ iff $(z = x$ or $z = y)]$

(UNION) *(Axiom of Unions)* Unions of sets exist:

For all sets $\mathcal{A}$ there is a set B such that $x \in B$ iff $\exists A \in \mathcal{A}$ s.t. $x \in A$

(POW) *(Power Set Axiom)* Every set has a power set:

For all sets A there is a set $\mathcal{P}$ such that $\forall B, B \in \mathcal{P}$ iff $(\forall x \in B, x \in A)$

Zermelo-Fraenkel Axioms

(SEP) *(Separation Axiom)* If A is a set and $Q(x)$ is a logical statement which depends on x then the subset of all elements of A satisfying $Q(x)$ exists:

$\forall A$ and statements $Q(x)$, $\exists B$ s.t. $\forall x, x \in B$ iff $x \in A$ and $Q(x)$ is true.

(REP) *(Replacement Axiom)* If A is a set and f is a function then $B = f(A)$ is a set:

For any set A and any statement $Q(x, y)$
$(\forall x \in A, \exists! y$ s.t. Q(x,y) is true) implies
$(\exists B$ s.t. $y \in B$ iff $\exists x \in A$ s.t. $Q(x, y)$ is true)

(EMP) *(Existence of an empty set)* There is a set which has no elements:

There exists E s.t. $\forall x, x \notin E$

Before we give last three axioms of ZFC need notation:

Notational Shortcuts

1. $\varnothing$ is set given by (EMP). (EXT) implies uniqueness.
2. $\{x, y\}$ is shorthand for the set given by (PAIR).
3. $\{x\}$ is shorthand for $\{x, x\}$.
4. $\{x \in A | Q(x)\}$ is shorthand for set given by (SEP) applied to set A and statement $Q(x)$.
5. $A \subset B$ is shorthand for statement $\forall x, (x \in A$ implies $x \in B)$.
6. $A \cup B$ is shorthand for set given by (UNION) applied to $\mathcal{A} = \{A, B\}$.
7. $\bigcup \mathcal{A}$ is set given by (UNION) applied to $\mathcal{A}$.
8. $A \cap B$ is shorthand for set $\{x \in A | x \in B\}$.
9. If $\mathcal{A} \neq \varnothing$ then let $A \in \mathcal{A}$. $\bigcap \mathcal{A}$ is shorthand for $\{x \in A | \forall B \in \mathcal{A}, x \in B\}$.
10. $\mathcal{P}(A)$ is set given by (POW).

Zermelo-Fraenkel Axioms

(REG) *(Axiom of Regularity)* Every nonempty set A contains an element x with $x \cap A = \varnothing$.

$$\text{For all sets } A \text{ if } A \neq \varnothing \text{ then}$$
$$\exists x \in A \text{ s.t. } (\nexists y \text{ s.t. } (y \in x) \text{ and } (y \in A))$$

(INF) *(Axiom of Infinity)* There is an infinite set and it contains $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}, \cdots$.

$$\text{There exists } S \text{ s.t. } \varnothing \in S \text{ and } \forall x \in S, x \cup \{x\} \in S.$$

(AC) *(Axiom of Choice)* Given a set $\mathcal{A}$ of nonempty disjoint sets there is a set C which contains exactly one element from each set in $\mathcal{A}$.

$$\text{For all } \mathcal{A} \text{ s.t. } (\varnothing \notin \mathcal{A}) \text{ and } (\forall A, B \in \mathcal{A}, A \neq B \text{ implies } A \cap B = \varnothing)$$
$$\exists C \text{ s.t. } \forall A \in \mathcal{A}, \exists! x \in C \text{ s.t. } x \in C \cap A.$$