Math 5801
General Topology and Knot Theory

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August 22, 2012

Course Info

Textbook (required)
ISBN-10: 0131816292

Reading for Friday
Chapter 1.1-1.3, pgs. 3-28

HW 1 for Monday
- Chapter 1.1: 3a, 4c
- Chapter 1.2: 2a-b, 4a-b, 5a
- Chapter 1.3: 2, 4, 11
Logical Statements

A *logical statement* (or simply a *statement*) is a mathematical assertion that is either *true* or *false*.

**Examples 1**

1. $1 + 2 = 3$. **true**
2. $e^7 = \pi$. **false**
3. $x = 2$ implies $x > 1$. **true**
4. (3 is even) and (6 is even). **false**
5. (3 is even) or (6 is even). **true**
6. It is not true that $1 + 1 = 2$. **false**
7. $x = y$. ?

In 3-6 above we used *logical operators*.

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Logical operators

If $P$ and $Q$ are statements then:

1. $(P$ and $Q)$ is true only if both $P$ is true and $Q$ is true.
2. $(P$ or $Q)$ is true unless both $P$ is false and $Q$ is false.
3. $(P$ is false).
4. $P$ implies $Q$) is true unless $P$ is true and $Q$ is false.
5. $(P$ iff $Q)$ is true only if truth values of $P$ and $Q$ agree.

Notation

1. $P \land Q$ means “$P$ and $Q$”
2. $P \lor Q$ means “$P$ or $Q$”
3. $\neg P$ means “it is not true that $P$”
4. $P \implies Q$ means “$P$ implies $Q$”
5. $P \iff Q$ means “$P$ if and only if $Q$” means “$P$ iff $Q$”
Logical Equivalence

For any statements $P$ and $Q$ the following statements are true:

1. $\neg(P \land Q) \iff (\neg P \lor \neg Q)$
2. $\neg(P \lor Q) \iff (\neg P \land \neg Q)$
3. $(P \implies Q) \iff (\neg P \lor Q)$
4. $(\neg P \iff Q) \iff ((P \land Q) \lor (\neg P \land \neg Q))$
5. $(P \implies Q) \iff (\neg Q \implies \neg P)$ (proof by contraposition)
6. $(P \implies (Q \land \neg Q)) \iff \neg P$ (proof by contradiction)

▶ Contrapositive of $P \implies Q$ is:

$$\neg Q \implies \neg P$$

▶ Converse of $P \implies Q$ is:

$$Q \implies P$$

Quantifiers

More complex statements can be constructed using quantifiers:

$\forall$ “for all” or “for every”

and

$\exists$ “there exists” or “there is some”

Examples 2

1. $\forall x \in \mathbb{R}, x^2 \geq 0$.
2. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $x + y = 0$.
3. $\forall n \in \mathbb{Z}, \exists y \in \mathbb{R}$ s.t. $y^2 = n$.
4. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$ (|$x - a|$ < $\delta \implies |f(x) - f(a)| < \varepsilon$).
Quantifiers

For any set $A$ and statement $P(x)$ which depends on $x$ we can construct new statements:

$$
\forall x \in A, P(x) \quad \text{“for all } x \in A, P(x) \text{ is true”}
$$

and

$$
\exists x \in A, P(x) \quad \text{“There is at least one } x \in A \text{ such that } P(x) \text{ is true”}
$$

$P(x)$ may itself be a quantified statement:

<table>
<thead>
<tr>
<th>Examples 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $x = y$</td>
</tr>
<tr>
<td>2. $\exists x \in \mathbb{R}, x = y$</td>
</tr>
<tr>
<td>3. $\forall y \in \mathbb{R}, \exists x \in \mathbb{R} \text{ s.t. } x = y \quad \text{true}$</td>
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Bound vs. Unbound Variables

− In the statement $\forall x \in A, P(x)$. The scope of $x$ is the statement $P(x)$.
− Within its scope $x$ has meaning and is said to be a bound variable.
− Otherwise $x$ is an unbound variable.
− It’s forbidden to quantify over a bound variable.

Example 4 (Scopes of variables)

$$
\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } x + y = 0.
$$
Bound vs. Unbound Variables

Common source of **nonsense** is problems with variable scope.

### Bound variables

1. $\forall x \in \mathbb{R}, \exists x \in \mathbb{R}$ s.t. $x = x$. **nonsense**
2. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $x = x$. **true**
3. $(\forall x \in \mathbb{R}, x = x)$ and $(x = 5)$. **?**
4. $\forall x \in \mathbb{R}, (x = x$ and $x = 5)$. **false**

In proofs we bind variables using words like “suppose” and “let” as well as “for all” and “there exists.”

**Example 5 (Proof that $\sqrt{2}$ is irrational)**

Suppose $x \in \mathbb{Q}$ and $x^2 = 2$. Then there exist $p, q \in \mathbb{Z}$ with $q \neq 0$ and $p$ and $q$ relatively prime s.t. $x = \frac{p}{q}$. Then $\left(\frac{p}{q}\right)^2 = 2$ so $p^2 = 2q^2$. Hence $p^2$ is even. It follows that $p$ is even so $p = 2k$ for some $k \in \mathbb{Z}$. Therefore $(2k)^2 = 2q^2$ so $2k^2 = q^2$. Thus $q^2$ is even which implies $q$ is even. Thus $2$ divides $p$ and $q$. It follows that $p$ and $q$ are not relatively prime. But $p$ and $q$ are relatively prime. CONTRADICTION.

Thus it cannot be simultaneously true that $x \in \mathbb{Q}$ and $x^2 = 2$. $\square$
Negating Quantified Statements

Negation of quantifiers

1. \( \neg (\exists x \in A, \text{ s.t. } P(x)) \) is logically equivalent to \( \forall x \in A, \neg P(x) \).
2. \( \neg (\forall x \in A, P(x)) \) is logically equivalent to \( \exists x \in A, \text{ s.t. } \neg P(x) \).

Fix a universe \( \mathcal{U} \).

- Everything in \( \mathcal{U} \) is a set.
- Any element of a set is itself a set.
- For example \( \varnothing \in \{ \varnothing, \{ \varnothing \} \} \)
- Any quantified statement (like \( \exists x, x \in x \)) is quantified over our entire universe \( \mathcal{U} \).
- We will assume \( \mathcal{U} \) satisfies the following collection of axioms (called ZFC)
Zermelo-Fraenkel Axioms

(EXT) (Extensionality Axiom) Sets are determined by their elements:
For all sets $A$ and $B$, $(A = B)$ iff $(\forall x, x \in A$ iff $x \in B)$

(PAIR) (Pair Set Axiom) The set $\{x, y\}$ exists:
For all $x$ and $y$, there is $A$ s.t. $\forall z, [z \in A$ iff $(z = x$ or $z = y)]$

(UNION) (Axiom of Unions) Unions of sets exist:
For all sets $\mathcal{A}$ there is a set $B$ such that $x \in B$ iff $\exists A \in \mathcal{A}$ s.t. $x \in A$

(POW) (Power Set Axiom) Every set has a power set:
For all sets $A$ there is a set $\mathcal{P}$ such that $\forall B, B \in \mathcal{P}$ iff $(\forall x \in B, x \in A)$

(SEP) (Separation Axiom) If $A$ is a set and $Q(x)$ is a logical statement which depends on $x$ then the subset of all elements of $A$ satisfying $Q(x)$ exists:
$\forall A$ and statements $Q(x)$, $\exists B$ s.t. $\forall x, x \in B$ iff $x \in A$ and $Q(x)$ is true.

(REP) (Replacement Axiom) If $A$ is a set and $f$ is a function then $B = f(A)$ is a set:
For any set $A$ and any statement $Q(x, y)$
$(\forall x \in A, \exists! y$ s.t. $Q(x, y)$ is true) implies
$(\exists B$ s.t. $y \in B$ iff $\exists x \in A$ s.t. $Q(x, y)$ is true)

(EMP) (Existence of an empty set) There is a set which has no elements:
There exists $E$ s.t. $\forall x, x \notin E$
Before we give last three axioms of ZFC need notation:

### Notational Shortcuts

1. $\emptyset$ is set given by (EMP). (EXT) implies uniqueness.
2. $\{x, y\}$ is shorthand for the set given by (PAIR).
3. $\{x\}$ is shorthand for $\{x, x\}$.
4. $\{x \in A | Q(x)\}$ is shorthand for set given by (SEP) applied to set $A$ and statement $Q(x)$.
5. $A \subseteq B$ is shorthand for statement $\forall x, (x \in A \text{ implies } x \in B)$.
6. $A \cup B$ is shorthand for set given by (UNION) applied to $A = \{A, B\}$.
7. $\bigcup \mathcal{A}$ is set given by (UNION) applied to $\mathcal{A}$.
8. $A \cap B$ is shorthand for set $\{x \in A | x \in B\}$.
9. If $\mathcal{A} \neq \emptyset$ then let $A \in \mathcal{A}$. $\bigcap \mathcal{A}$ is shorthand for $\{x \in A | \forall B \in \mathcal{A}, x \in B\}$.
10. $\mathcal{P}(A)$ is set given by (POW).

### Zermelo-Fraenkel Axioms

**REG** *(Axiom of Regularity)* Every nonempty set $A$ contains an element $x$ with $x \cap A = \emptyset$.

For all sets $A$ if $A \neq \emptyset$ then

$\exists x \in A \text{ s.t. } (\exists y \text{ s.t. } (y \in x) \text{ and } (y \in A))$

**INF** *(Axiom of Infinity)* There is an infinite set and it contains $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, $\cdots$.

There exists $S$ s.t. $\emptyset \in S$ and $\forall x \in S, x \cup \{x\} \in S$.

**AC** *(Axiom of Choice)* Given a set $\mathcal{A}$ of nonempty disjoint sets there is a set $C$ which contains exactly one element from each set in $\mathcal{A}$.

For all $\mathcal{A}$ s.t. $(\emptyset \notin \mathcal{A})$ and $(\forall A, B \in \mathcal{A}, A \neq B \text{ implies } A \cap B = \emptyset)$

$\exists C \text{ s.t. } \forall A \in \mathcal{A}, \exists! x \in C \text{ s.t. } x \in C \cap A$. 