Math 5801
General Topology and Knot Theory

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Course Info

Textbook (required)
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Reading for Monday
Chapter 1.4-1.6, pgs. 30-43

Reminder: HW 1 for Monday
- Chapter 1.1: 3a, 4c
- Chapter 1.2: 2a-b, 4a-b, 5a
- Chapter 1.3: 2, 4, 11
Fix a universe $\mathcal{U}$ with a relation $\in$.
Everything in $\mathcal{U}$ is a set.
Any element of a set is itself a set.
For example $\emptyset \in \{\emptyset, \{\emptyset\}\}$
Any quantified statement (like $\exists x, x \in x$) is quantified over our entire universe $\mathcal{U}$.
We will assume $\mathcal{U}$ satisfies the following collection of axioms (called ZFC)

Zermelo-Fraenkel Axioms

(EXT) *(Extensionality Axiom)* Sets are determined by their elements:

For all sets $A$ and $B$, $(A = B)$ iff $(\forall x, x \in A \text{ iff } x \in B)$

(PAIR) *(Pair Set Axiom)* The set $\{x, y\}$ exists:

For all $x$ and $y$, there is $A$ s.t. $\forall z, [z \in A \text{ iff } (z = x \text{ or } z = y)]$

(UNION) *(Axiom of Unions)* Unions of sets exist:

For all sets $A$ there is a set $B$ such that $x \in B$ iff $\exists A \in A \text{ s.t. } x \in A$

(POW) *(Power Set Axiom)* Every set has a power set:

For all sets $A$ there is a set $\mathcal{P}$ such that $\forall B, B \in \mathcal{P}$ iff $(\forall x \in B, x \in A)$
Zermelo-Fraenkel Axioms

(SEP) \textit{(Separation Axiom)} If \(A\) is a set and \(Q(x)\) is a logical statement which depends on \(x\) then the subset of all elements of \(A\) satisfying \(Q(x)\) exists:

\[\forall A \text{ and statements } Q(x), \exists B \text{ s.t. } \forall x, x \in B \text{ iff } x \in A \text{ and } Q(x) \text{ is true.}\]

(REP) \textit{(Replacement Axiom)} If \(A\) is a set and \(f\) is a function then \(B = f(A)\) is a set:

\[\text{For any set } A \text{ and any statement } Q(x, y) \]
\[\quad (\forall x \in A, \exists! y \text{ s.t. } Q(x, y) \text{ is true}) \implies\]
\[\quad (\exists B \text{ s.t. } y \in B \text{ iff } \exists x \in A \text{ s.t. } Q(x, y) \text{ is true})\]

(EMP) \textit{(Existence of an empty set)} There is a set which has no elements:

\[\text{There exists } E \text{ s.t. } \forall x, x \notin E\]

Before we give last three axioms of ZFC need notation:

\textbf{Definition 6} \((\emptyset, \{, \}, \{x \in A|Q(x)\}, \subset, \cap, \cup, \mathcal{P}(\cdot))\)

1. \(\emptyset\) is set given by (EMP). (EXT) implies uniqueness. (See Prop. 7 below)
2. \(\{x, y\}\) is shorthand for the set given by (PAIR).
3. \(\{x\}\) is shorthand for \(\{x, x\}\).
4. \(\{x \in A|Q(x)\}\) is shorthand for set given by (SEP) applied to set \(A\) and statement \(Q(x)\).
5. \(A \subset B\) is shorthand for statement \(\forall x, (x \in A \implies x \in B)\).
6. \(A \cup B\) is shorthand for set given by (UNION) applied to \(A = \{A, B\}\).
7. \(\bigcup A\) is set given by (UNION) applied to \(A\).
8. \(A \cap B\) is shorthand for set \(\{x \in A|x \in B\}\).
9. If \(A \neq \emptyset\) then let \(A \in A\). \(\bigcap A\) is shorthand for \(\{x \in A|\forall B \in A, x \in B\}\).
10. \(\mathcal{P}(A)\) is set given by (POW).
Zermelo-Fraenkel Axioms

(REG) (Axiom of Regularity) Every nonempty set $A$ contains an element $x$ with $x \cap A = \emptyset$.

For all sets $A$ if $A \not= \emptyset$ then
\[
\exists x \in A \text{ s.t. } (\nexists y \text{ s.t. } (y \in x) \text{ and } (y \in A))
\]

(INF) (Axiom of Infinity) There is an infinite set and it contains $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, $\ldots$.

There exists $S$ s.t. \(\emptyset \in S\) and \(\forall x \in S, x \cup \{x\} \in S\).

(AC) (Axiom of Choice) Given a set $\mathcal{A}$ of nonempty disjoint sets there is a set $C$ which contains exactly one element from each set in $\mathcal{A}$.

For all $\mathcal{A}$ s.t. \((\emptyset \not\in \mathcal{A})\) and \((\forall A, B \in \mathcal{A}, A \not= B \text{ implies } A \cap B = \emptyset)\)
\[
\exists C \text{ s.t. } \forall A \in \mathcal{A}, \exists! x \in C \text{ s.t. } x \in C \cap A.
\]

That’s all!

IF there is a universe \(\mathcal{U}\) with a relation \(\in\) satifying:

\[
\text{ZFC} = \left\{ \text{ (EXT), (PAIR), (UNION), (POW), (SEP) } \right\} \\
\left\{ \text{ (REP), (EMP), (REG), (INF), (AC) } \right\}
\]

then we can do all modern math in \(\mathcal{U}\).

BUT! Gödel’s Second Incompleteness Theorem says

Existence of such a universe \(\mathcal{U}\) is unprovable!

So we assume \(\mathcal{U}\) exists and move on.

Someday someone might prove inconsistency of ZFC.
Basic Properties from ZFC

- Forget everything you know about set theory.
- Can only talk about sets that exist due to ZFC axioms.
- Note: We can prove existence of some sets without Axiom of Choice (AC). Such sets are *constructible*.
- Using (AC) is fine but sets whose existence depends on (AC) are “inaccessible”.

Proposition 7 (Empty set is unique)

Let $\emptyset$ denote the set given by (EMP). Then if the set $A$ satisfies same properties as given in (EMP) then $A = \emptyset$.

Proof.

- By (EMP) there exists a set $E$ such that $\forall x, x \notin E$. Call it $\emptyset$.
- Suppose there is a set $A$ such that $\forall x, x \notin A$.
- Then $\forall x, (x \notin A)$ and $(x \notin \emptyset)$.
- Then $\forall x, \lnot (x \in A)$ and $\lnot (x \in \emptyset)$.
- Then $\forall x, (\lnot (x \in A) \text{ and } \lnot (x \in \emptyset))$ or $((x \in A) \text{ and } (x \in \emptyset))$.
- Then $\forall x, (x \in A) \text{ iff } (x \in \emptyset)$.
- So by (EXT) $A = \emptyset$. 

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Basic Properties from ZFC

**Proposition 8 (Subsets of the empty set are empty)**

\[ A \subseteq \emptyset \iff A = \emptyset. \]

**Proof.**

- \( A = \emptyset \)
- \( \iff \forall x, (x \notin A) \)
- \( \iff \forall x, \neg (x \in A) \)
- \( \iff \forall x, (x \in A) \implies (x \in \emptyset) \)
- \( \iff A \subseteq \emptyset. \)

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**Proposition 9**

\[ \mathcal{P}(\emptyset) = \{\emptyset\}. \]

**Proof.**

- By definition \( \mathcal{P}(\emptyset) \) has property given in (POW). Namely
  \[ \forall B, B \in \mathcal{P}(\emptyset) \iff \forall x \in B, x \in \emptyset. \]
- By definition \( \{\emptyset\} = \{\emptyset, \emptyset\} \) which by (PAIR) satisfies
  \[ \forall B, B \in \{\emptyset, \emptyset\} \iff (B = \emptyset) \text{ or } (B = \emptyset). \]
  Thus \( \forall B, B \in \emptyset \iff B = \emptyset. \)
- \( B \in \mathcal{P}(\emptyset) \iff \forall x \in B, x \in \emptyset \iff B \subseteq \emptyset \iff B = \emptyset \iff B \in \emptyset. \)
- Thus \( \forall B, B \in \mathcal{P}(\emptyset) \iff B \in \emptyset. \)
- Thus by (EXT) we have \( \mathcal{P}(\emptyset) = \{\emptyset\}. \)
Basic Properties from ZFC

Proposition 10 (Miscellaneous properties of sets)

If $A$, $B$ and $C$ are sets then:

1. $A \subset A$. (Reflexivity)
2. $A = B$ iff ($A \subset B$ and $B \subset A$).
3. $(A \subset B$ and $B \subset C$) implies $A \subset C$. (Transitivity)
4. $A \subset A \cup B$.
5. $A \cap B \subset A$.
6. $\emptyset \subset A$.
7. $A \cup \emptyset = A$.
8. $A \cap \emptyset = \emptyset$.
9. $\bigcup \emptyset = \emptyset$.
10. $B \in \mathcal{P}(A)$ iff $B \subset A$.

Proposition 11 (More properties of sets)

1. $x \in \{y\}$ implies $x = y$.
2. $\{a, b\} = \{x, y\}$ implies (($a = x$ and $b = y$) or ($a = y$ and $b = x$)).
Basic Properties from ZFC

Proposition 12
For any set $A$ we have $A \notin A$.

Proof.
- Let $A$ be a set.
- Suppose $A \in A$.
  - By (PAIR) the set $\{A\}$ exists.
  - $A \in A$ and $A \in \{A\}$ so $A \cap \{A\} \neq \emptyset$.
  - By (REG) there is $x \in \{A\}$ such that $x \cap \{A\} = \emptyset$.
  - But $x \in \{A\}$ implies $x = A$ implies $x \cap \{A\} = A \cap \{A\} \neq \emptyset$.
  - CONTRADICTION.
- so $A \notin A$.

Ordered Pairs in ZFC

Definition 13 (Ordered pair)
$$(x, y) = \{x, \{x, y\}\}.$$

Proposition 14
$$(x, y) = (z, w) \text{ iff } x = z \text{ and } y = w.$$
Ordered Pairs in ZFC

Proof of Prop. 14 (continued).

\(\iff\)
- Suppose \((x, y) = (z, w)\).
- Then \(\{x, \{x, y\}\} = \{z, \{z, w\}\}\).
- Then \((x = z \text{ and } \{x, y\} = \{z, w\})\) or \((x = \{z, w\} \text{ and } \{x, y\} = z)\).
- **CASE I:** Suppose \(x = z\) and \(\{x, y\} = \{z, w\}\).
  - Then \(x = z\) and \(\{(x = z \text{ and } y = w) \text{ or } (x = w \text{ and } y = z)\}\).
  - If \(x = z \text{ and } (x = z \text{ and } y = w)\) then we are done.
  - If \(x = z \text{ and } (x = w \text{ and } y = z)\) then \(w = x = z = y\) so \((x = z \text{ and } y = w)\) and we are done.
- **CASE II:** Suppose \(x = z\) and \(x = \{z, w\}\) and \(\{x, y\} = z\).
  - Then \(x = \{x, y\}\).
  - But then \(x\) is a member of itself. CONTRADICTION (Prop. 12)

The Cartesian Product

- We’d like to define \(A \times B = \{(a, b) | a \in A \text{ and } b \in B\}\).
- Suppose \(a \in A \text{ and } b \in B\).
- By definition \((a, b) = \{a, \{a, b\}\}\).
- In order to use (SEP) we need a set which contains each \((a, b)\).
- \(a \in A \text{ and } \{a, b\} \in \mathcal{P}(A \cup B)\).
- So \(\{a, \{a, b\}\} \subset A \cup \mathcal{P}(A \cup B)\).
- So \(\{a, \{a, b\}\} \in \mathcal{P}(A \cup \mathcal{P}(A \cup B))\).

**Definition 15 (Cartesian Product)**

\[
A \times B = \{ C \in \mathcal{P}(A \cup \mathcal{P}(A \cup B)) | \exists a \in A, b \in B \text{ s.t. } C = (a, b) \}. \\
\]
or more informally,

\[
A \times B = \{(a, b) | a \in A \text{ and } b \in B\}. \\
\]
Functions

**Definition 16 (Function)**

- Let $A$ and $B$ be sets. A *function* $f : A \to B$ is a pair $f = (G, B)$ where $G \subset A \times B$ and for all $a \in A$ there is a unique $b \in B$ such that $(a, b) \in G$.
- If $(a, b) \in G$ we write “$f(a) = b$.”
- The set $\text{dom}(f) = A$ is called the *domain* of $f$.
- The set $\text{codom}(f) = B$ is called the *codomain* or *range* of $f$.
- The set $\text{graph}(f) = G$ is called the *graph* of $f$.

**Observations:**

- Let $f$ be a function. Then $a \in \text{dom}(f)$ iff $\exists b \in B \text{ s.t. } (a, b) \in \text{graph}(f)$.
- If $f$ and $g$ are functions and $f = g$ then $\text{graph}(f) = \text{graph}(g)$, $\text{codom}(f) = \text{codom}(g)$. Moreover $\text{dom}(f) = \text{dom}(g)$.