Math 5801
General Topology and Knot Theory

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Course Info

Textbook (required)

J. R. Munkres, *Topology (2nd Edition)*, Prentice Hall, Englewood Cliffs, NJ, 2000.

ISBN-10: 0131816292

Reading for Monday

Chapter 1.4-1.6, pgs. 30-43

Reminder: HW 1 for Monday

- Chapter 1.1: 3a, 4c
- Chapter 1.2: 2a-b, 4a-b, 5a
- Chapter 1.3: 2, 4, 11

- Fix a universe $\mathfrak{U}$ with a relation $\in$.
- Everything in $\mathfrak{U}$ is a set.
- Any element of a set is itself a set.
- For example $\varnothing \in \{\varnothing, \{\{\varnothing\}\}\}$
- Any quantified statement (like $\exists x, x \in x$) is quantified over our entire universe $\mathfrak{U}$.
- We will assume $\mathfrak{U}$ satisfies the following collection of axioms (called ZFC)

Zermelo-Fraenkel Axioms

(EXT) *(Extensionality Axiom)* Sets are determined by their elements:

For all sets A and B, $(A = B)$ iff $(\forall x,\ x \in A$ iff $x \in B)$

(PAIR) *(Pair Set Axiom)* The set $\{x, y\}$ exists:

For all x and y, there is A s.t. $\forall z, [z \in A$ iff $(z = x$ or $z = y)]$

(UNION) *(Axiom of Unions)* Unions of sets exist:

For all sets $\mathcal{A}$ there is a set B such that $x \in B$ iff $\exists A \in \mathcal{A}$ s.t. $x \in A$

(POW) *(Power Set Axiom)* Every set has a power set:

For all sets A there is a set $\mathcal{P}$ such that $\forall B, B \in \mathcal{P}$ iff $(\forall x \in B, x \in A)$

Zermelo-Fraenkel Axioms

(SEP) *(Separation Axiom)* If A is a set and $Q(x)$ is a logical statement which depends on x then the subset of all elements of A satisfying $Q(x)$ exists:

$\forall A$ and statements $Q(x)$, $\exists B$ s.t. $\forall x, x \in B$ iff $x \in A$ and $Q(x)$ is true.

(REP) *(Replacement Axiom)* If A is a set and f is a function then $B = f(A)$ is a set:

For any set A and any statement $Q(x, y)$
($\forall x \in A, \exists! y$ s.t. Q(x,y) is true) implies
($\exists B$ s.t. $y \in B$ iff $\exists x \in A$ s.t. $Q(x, y)$ is true)

(EMP) *(Existence of an empty set)* There is a set which has no elements:

There exists E s.t. $\forall x, x \notin E$

Before we give last three axioms of ZFC need notation:

Definition 6 ($\varnothing$, $\{,\}$, $\{x \in A | Q(x)\}$, $\subset$, $\cap$, $\cup$, $\mathcal{P}()$)

1. $\varnothing$ is set given by (EMP). (EXT) implies uniqueness. (See Prop. 7 below)
2. $\{x, y\}$ is shorthand for the set given by (PAIR).
3. $\{x\}$ is shorthand for $\{x, x\}$.
4. $\{x \in A | Q(x)\}$ is shorthand for set given by (SEP) applied to set A and statement $Q(x)$.
5. $A \subset B$ is shorthand for statement $\forall x, (x \in A$ implies $x \in B)$.
6. $A \cup B$ is shorthand for set given by (UNION) applied to $\mathcal{A} = \{A, B\}$.
7. $\bigcup \mathcal{A}$ is set given by (UNION) applied to $\mathcal{A}$.
8. $A \cap B$ is shorthand for set $\{x \in A | x \in B\}$.
9. If $\mathcal{A} \neq \varnothing$ then let $A \in \mathcal{A}$. $\bigcap \mathcal{A}$ is shorthand for $\{x \in A | \forall B \in \mathcal{A}, x \in B\}$.
10. $\mathcal{P}(A)$ is set given by (POW).

Zermelo-Fraenkel Axioms

(REG) *(Axiom of Regularity)* Every nonempty set A contains an element x with $x \cap A = \varnothing$.

$$\text{For all sets } A \text{ if } A \neq \varnothing \text{ then}$$
$$\exists x \in A \text{ s.t. } (\nexists y \text{ s.t. } (y \in x) \text{ and } (y \in A))$$

(INF) *(Axiom of Infinity)* There is an infinite set and it contains $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}, \cdots$.

$$\text{There exists } S \text{ s.t. } \varnothing \in S \text{ and } \forall x \in S, x \cup \{x\} \in S.$$

(AC) *(Axiom of Choice)* Given a set $\mathcal{A}$ of nonempty disjoint sets there is a set C which contains exactly one element from each set in $\mathcal{A}$.

$$\text{For all } \mathcal{A} \text{ s.t. } (\varnothing \notin \mathcal{A}) \text{ and } (\forall A, B \in \mathcal{A}, A \neq B \text{ implies } A \cap B = \varnothing)$$
$$\exists C \text{ s.t. } \forall A \in \mathcal{A}, \exists! x \in C \text{ s.t. } x \in C \cap A.$$

Zermelo-Fraenkel Axioms

- That's all!
- IF there is a universe $\mathfrak{U}$ with a relation $\in$ satifying:
- $$\text{ZFC} = \left\{ \begin{array}{c} \text{(EXT), (PAIR), (UNION), (POW), (SEP)} \\ \text{(REP), (EMP), (REG), (INF), (AC)} \end{array} \right\}$$
- then we can do all modern math in $\mathfrak{U}$.
- BUT! Gödel's Second Incompleteness Theorem says
- Existence of such a universe $\mathfrak{U}$ is unprovable!
- So we assume $\mathfrak{U}$ exists and move on.
- Someday someone might prove inconsistency of ZFC.

Basic Properties from ZFC

- Forget everything you know about set theory.
- Can only talk about sets that exist due to ZFC axioms.
- it Note: We can prove existence of some sets without Axiom of Choice (AC). Such sets are *constructible*.
- Using (AC) is fine but sets whose existence depends on (AC) are "inaccessible".

Basic Properties from ZFC

Proposition 7 (Empty set is unique)

Let $\varnothing$ denote the set given by (EMP). Then if the set A satisfies same properties as given in (EMP) then $A = \varnothing$.

Proof.

- By (EMP) there exists a set E such that $\forall x, x \notin E$. Call it $\varnothing$.
- Suppose there is A s.t. $\forall x, x \notin A$.
- Then $\forall x, (x \notin A)$ and $(x \notin \varnothing)$.
- Then $\forall x, \neg(x \in A)$ and $\neg(x \in \varnothing)$.
- Then $\forall x, \big(\neg(x \in A)$ and $\neg(x \in \varnothing)\big)$ or $\big((x \in A)$ and $(x \in \varnothing)\big)$.
- Then $\forall x, (x \in A)$ iff $(x \in \varnothing)$.
- So by (EXT) $A = \varnothing$.

$\square$

Basic Properties from ZFC

Proposition 8 (Subsets of the empty set are empty)

$A \subset \varnothing$ *iff* $A = \varnothing$.

Proof.

- $A = \varnothing$
- iff $\forall x, (x \notin A)$
- iff $\forall x, \neg(x \in A)$
- iff $\forall x, (x \in A)$ implies $(x \in \varnothing)$
- iff $A \subset \varnothing$.

$\square$

Basic Properties from ZFC

Proposition 9

$\mathcal{P}(\varnothing) = \{\varnothing\}$.

Proof.

- By definition $\mathcal{P}(\varnothing)$ has property given in (POW). Namely

$$\forall B, B \in \mathcal{P}(\varnothing) \text{ iff } \forall x \in B, x \in \varnothing.$$

- By definition $\{\varnothing\} = \{\varnothing, \varnothing\}$ which by (PAIR) satisfies

$$\forall B, B \in \{\varnothing, \varnothing\} \text{ iff } (B = \varnothing) \text{ or } (B = \varnothing).$$
$$\text{Thus } \forall B, B \in \{\varnothing\} \text{ iff } B = \varnothing.$$

- $B \in \mathcal{P}(\varnothing)$ iff $\forall x \in B, x \in \varnothing$ iff $B \subset \varnothing$ $\underbrace{\text{iff}}_{\text{Prop. 8}}$ $B = \varnothing$ iff $B \in \{\varnothing\}$.
- Thus $\forall B, B \in \mathcal{P}(\varnothing)$ iff $B \in \{\varnothing\}$.
- Thus by (EXT) we have $\mathcal{P}(\varnothing) = \{\varnothing\}$. $\square$

Basic Properties from ZFC

Proposition 10 (Miscellaneous properties of sets)

If A, B and C are sets then:

1. $A \subset A$. *(Reflexivity)*
2. $A = B$ *iff* $(A \subset B$ *and* $B \subset A)$.
3. $(A \subset B$ *and* $B \subset C)$ *implies* $A \subset C$. *(Transitivity)*
4. $A \subset A \cup B$.
5. $A \cap B \subset A$.
6. $\varnothing \subset A$.
7. $A \cup \varnothing = A$.
8. $A \cap \varnothing = \varnothing$.
9. $\bigcup \varnothing = \varnothing$.
10. $B \in \mathcal{P}(A)$ *iff* $B \subset A$.

Basic Properties from ZFC

Proposition 11 (More properties of sets)

1. $x \in \{y\}$ *implies* $x = y$.
2. $\{a, b\} = \{x, y\}$ *implies* $((a = x$ *and* $b = y)$ *or* $(a = y$ *and* $b = x))$.

Basic Properties from ZFC

Proposition 12

For any set A we have $A \notin A$.

Proof.

- Let A be a set.
- Suppose $A \in A$.
  - By (PAIR) the set $\{A\}$ exists.
  - $A \in A$ and $A \in \{A\}$ so $A \cap \{A\} \neq \varnothing$.
  - By (REG) there is $x \in \{A\}$ such that $x \cap \{A\} = \varnothing$.
  - But $x \in \{A\}$ $\underbrace{\text{implies}}_{\text{Prop. 11(1)}}$ $x = A$ implies $x \cap \{A\} = A \cap \{A\} \neq \varnothing$.
  - CONTRADICTION.
- so $A \notin A$.

$\square$

Ordered Pairs in ZFC

Definition 13 (Ordered pair)

$(x, y) = \{x, \{x, y\}\}$.

Proposition 14

$(x, y) = (z, w)$ *iff* $x = z$ *and* $y = w$.

Proof.

$\Rightarrow$

- Suppose $x = z$ and $y = w$.
- Then $\{x, \{x, y\}\} = \{z, \{z, w\}\}$.
- Thus $(x, y) = (z, w)$.

Ordered Pairs in ZFC

Proof of Prop. 14 *(continued)*.

$\Leftarrow$

- Suppose $(x, y) = (z, w)$.
- Then $\{x, \{x, y\}\} = \{z, \{z, w\}\}$.
- Then ($x = z$ and $\{x, y\} = \{z, w\}$) or ($x = \{z, w\}$ and $\{x, y\} = z$).
- CASE I: Suppose $x = z$ and $\{x, y\} = \{z, w\}$.
  - Then $x = z$ and (($x = z$ and $y = w$) or ($x = w$ and $y = z$)).
  - If $x = z$ and ($x = z$ and $y = w$) then we are done.
  - If $x = z$ and ($x = w$ and $y = z$) then $w = x = z = y$ so ($x = z$ and $y = w$) and we are done.
- CASE II: Suppose $x = z$ and $x = \{z, w\}$ and $\{x, y\} = z$.
  - Then $x = \{x, y\}$.
  - But then x is a member of itself. CONTRADICTION (Prop. 12)

$\square$

The Cartesian Product

- We'd like to define $A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$.
- Suppose $a \in A$ and $b \in B$.
- By definition $(a, b) = \{a, \{a, b\}\}$.
- In order to use (SEP) we need a set which contains each (a, b).
- $a \in A$ and $\{a, b\} \in \mathcal{P}(A \cup B)$.
- So $\{a, \{a, b\}\} \subset A \cup \mathcal{P}(A \cup B)$.
- So $\{a, \{a, b\}\} \in \mathcal{P}(A \cup \mathcal{P}(A \cup B))$.

Definition 15 (Cartesian Product)

$$A \times B = \left\{ C \in \mathcal{P}(A \cup \mathcal{P}(A \cup B)) \,\middle|\, \exists a \in A, b \in B \text{ s.t. } C = (a, b) \right\}.$$

or more informally,

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

Functions

Definition 16 (Function)

- Let A and B be sets. A *function* $f : A \to B$ is a pair $f = (G, B)$ where $G \subset A \times B$ and for all $a \in A$ there is a unique $b \in B$ such that $(a, b) \in G$.
- If $(a, b) \in G$ we write "$f(a) = b$."
- The set $\mathrm{dom}(f) = A$ is called the *domain* of f.
- The set $\mathrm{codom}(f) = B$ is called the *codomain* or *range* of f.
- The set $\mathrm{graph}(f) = G$ is called the *graph* of f.

Observations:

- Let f be a function. Then $a \in \mathrm{dom}(f)$ iff $\exists b \in B$ s.t. $(a, b) \in \mathrm{graph}(f)$.
- If f and g are functions and $f = g$ then $\mathrm{graph}(f) = \mathrm{graph}(g)$, $\mathrm{codom}(f) = \mathrm{codom}(g)$. Moreover $\mathrm{dom}(f) = \mathrm{dom}(g)$.