Math 5801  
General Topology and Knot Theory

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Course Info

Reading for Friday, September 7
Chapter 2.14-2.15, pgs. 84-88

HW 3 for Monday, September 10
- Chapter 1.9: 1, 2b, 6a
- Chapter 1.10: 1, 2a-b
## Axiom of Choice

### Axiom of Choice (AC)

Let \( A \) be a set of disjoint sets. Then there is a choice set \( C \) such that for all \( A \in A \) there is a unique element \( a \in A \cap C \).

Propositions 69, 70, 71 and 74 below are all equivalent to (AC):

### Proposition 69 (Choice functions exist)

Let \( B \) be a set of nonempty sets. Then there is a choice function

\[
c : B \to \bigcup_{B \in B} B
\]

such that for all \( B \in B \) we have \( c(B) \in B \).

### Proposition 70 (Every set can be well-ordered)

For every set \( A \) there is an order relation \( < \) for \( A \) which is a well-ordering.

### Proposition 71 (Maximum Principle)

Every set \( A \) with a partial order contains a maximal subset \( B \subset A \) such that the restriction of the partial order of \( A \) to \( B \) is an order relation for \( B \).

### Definition 72 (Partial Order)

Let \( A \) be a set. A **partial order** on \( A \) is a relation \( C \) on \( A \) satisfying:

1. (Nonrefexivity) For all \( x \in A \), \( \neg xCx \).

2. (Transitivity) If \( xCy \) and \( yCz \) then \( xCz \).

We’ll usually write \( x \prec y \) or \( y \succ x \) instead of \( xCy \).
Axiom of Choice

Example 73

If $B$ is a set then proper containment gives a partial order on $A^\mathcal{P}(B)$.

- Let $B = \{0, 1, 2\}$.
- $\{0, 2\} \subsetneq \{0, 1, 2\}$
- $\{0, 2\}$ and $\{0, 1\}$ are incomparable.
- A maximal subset of $\mathcal{P}(B)$ which is totally ordered is
  $T = \{\emptyset, \{1\}, \{1, 2\}, \{0, 1, 2\}\}$
- $X = \{\emptyset, \{1, 2\}\}$ is also totally ordered but not maximal since $X \subset B$.
- Another maximal subset of $\mathcal{P}(B)$ which is totally ordered is
  $C = \{\emptyset, \{2\}, \{0, 2\}, \{0, 1, 2\}\}$

Proposition 74 (Zorn’s Lemma)

Let $A$ be a set with a partial order. If every totally ordered subset of $A$ has a maximal element then $A$ has a maximal element.

Example 75

If $B$ is a set then proper containment gives a partial order on $A = \mathcal{P}(B)$.

- $X = \{\emptyset, \{1\}\}$ is also totally ordered and has a maximal elt. $\{1, 2\}$.
- $A$ has a maximal element. $\{0, 1, 2\}$.
- Set of proper subsets of $B$ has 3 max elements ($\{1, 2\}, \{0, 2\}$, and $\{0, 1\}$).
Definition 76 (Topological Space)

A **topological space** is a pair \((X, T)\) where \(X\) is a set and \(T \subseteq \mathcal{P}(X)\) is a collection of subsets satisfying:

1. \(\emptyset, X \in T\)
2. \(A \subseteq T\) implies \(\bigcup A \in T\).
3. \(A, B \in T\) implies \(A \cap B \in T\).

If \(A \in T\) we say \(A\) is an **open set**.
If \(X - A \in T\) we say \(A\) is a **closed set**.

- Subsets of \(X\) can be open, closed, both or neither.
- Rule (2) says **arbitrary unions** of open sets are open.
- Rule (3) implies **finite intersections** of open sets are open.

Examples 77

Let \(X = \{1, 2, 3\}\). Decide if the following choices for \(T\) make \((X, T)\) a topological space.

- \(T_1 = \{\emptyset, \{1, 2, 3\}\}\). Is a topology.
- \(T_2 = \mathcal{P}(X)\). Is a topology.
- \(T_3 = \{\emptyset, \{3\}, \{2, 3\}, \{1, 2, 3\}\}\). Is a topology.
- \(T_4 = \{\emptyset, \{1\}, \{3\}, \{1, 2, 3\}\}\). Not a topology.

Definition 78 (A few important topologies)

Let \(X\) be a set. The following are always topologies on \(X\):

1. \(T_t = \{\emptyset, X\}\) is the **trivial topology** on \(X\).
2. \(T_d = \mathcal{P}(X)\) is the **discrete topology** on \(X\).
3. \(T_f = \{\emptyset\} \cup \{X - F \mid F\) a finite subset of \(X\}\) is the **finite complement topology** on \(X\).
Topology

Definition 79 (Comparable topologies)
Let $X$ be a set. Let $\mathcal{T}, \mathcal{T}'$ be two topologies on $X$.
1. $\mathcal{T}, \mathcal{T}'$ are comparable if $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$.
2. $\mathcal{T}$ is coarser than $\mathcal{T}'$ if $\mathcal{T} \subset \mathcal{T}'$.
3. $\mathcal{T}$ is finer than $\mathcal{T}'$ if $\mathcal{T} \supset \mathcal{T}'$.

The set $\{\mathcal{T} \subset \mathcal{P}(X) | \mathcal{T} \text{ is a topology on } X\}$ is a poset.

Examples 80
1. $\mathcal{T}_d = \mathcal{P}(X)$ is finest topology on $X$.
2. $\mathcal{T}_t = \{\emptyset, X\}$ is coarsest topology on $X$.
3. $\emptyset, \{1\}, \{1, 2, 3\}$ and $\emptyset, \{1, 2\}, \{1, 2, 3\}$ are incomparable topologies on $X = \{1, 2, 3\}$.

Basis for a Topology

Definition 81 (Basis for a topology)
Let $X$ be a set. Let $\mathcal{B} \subset \mathcal{P}(X)$ satisfy:
1. For all $x \in X$ there is $B \in \mathcal{B}$ such that $x \in B$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Then $\mathcal{B}$ is called a basis for a topology on $X$.

Let $\mathcal{B}$ be a basis for a topology on $X$. Let

$$\mathcal{T} = \{U \in \mathcal{P}(X) | \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U\}$$

$\mathcal{T}$ is the topology generated by $\mathcal{B}$. 
Basis for a Topology

Proposition 82

Let $X$ be a set and $\mathcal{B} \subset \mathcal{P}(X)$ be a basis for a topology on $X$. Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. Then $\mathcal{T}$ is a topology on $X$.

Proof.

$X$ be a set and $\mathcal{B} \subset \mathcal{P}(X)$ be a basis for a topology on $X$.

- **Claim:** $\emptyset \in \mathcal{T}$
  - $\forall x \in \emptyset$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset \emptyset$.
- **Claim:** $X \in \mathcal{T}$
  - By Basis Rule (1) $\forall x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset X$.
- **Claim:** $\mathcal{A} \subset \mathcal{T} \Rightarrow \bigcup \mathcal{A} \in \mathcal{T}$.
  - Suppose $\mathcal{A} \subset \mathcal{T}$.
  - Suppose $x \in \bigcup \mathcal{A}$
  - Then $\exists A \in \mathcal{A}$ s.t. $x \in A$.
  - $A \in \mathcal{A}$ so $A \in \mathcal{T}$.
  - Hence there is $B \in \mathcal{B}$ such that $x \in B \subset A$.
  - Thus $\bigcup \mathcal{A} \in \mathcal{T}$.

- **Claim:** $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$.
  - Suppose $U, V \in \mathcal{T}$.
  - Suppose $x \in U \cap V$.
  - Then $x \in U$ and $x \in V$.
  - So $\exists B_1 \in \mathcal{B}$ s.t. $x \in B_1 \subset U$.
  - And $\exists B_2 \in \mathcal{B}$ s.t. $x \in B_2 \subset V$.
  - By Basis Rule (2) $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2 \subset U \cap V$.
  - So $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset U \cap V$.
  - Thus $U \cap V \in \mathcal{T}$.

- We have shown that $\mathcal{T}$ satisfies Rules (1)-(3) for a topology.
Proposition 83 (Alt. char. of topology generated by $\mathcal{B}$)

Let $X$ be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$ be a basis for a topology on $X$. Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. Then

$$\mathcal{T} = \left\{ \bigcup \mathcal{U} \mid \mathcal{U} \subseteq \mathcal{B} \right\}$$

Proof.

- **Claim I:** $\mathcal{B} \subseteq \mathcal{T}$
  - Let $B \in \mathcal{B}$.
  - Then $\forall x \in B, B \in \mathcal{B}$ and $x \in B \subseteq B$.
  - Hence $B \in \mathcal{T}$.

Proof of Prop. 83 *(continued)*.

- **Claim II:** $\left\{ \bigcup \mathcal{U} \mid \mathcal{U} \subseteq \mathcal{B} \right\} \subseteq \mathcal{T}$.
  - Suppose $\mathcal{U} \subseteq \mathcal{B}$.
  - Then by Claim I we have $\mathcal{U} \subseteq \mathcal{T}$.
  - By Proposition 82 $\mathcal{T}$ is a topology so by Top. Rule (2) $\bigcup \mathcal{U} \in \mathcal{T}$.

- **Claim III:** $\mathcal{T} \subseteq \left\{ \bigcup \mathcal{U} \mid \mathcal{U} \subseteq \mathcal{B} \right\}$
  - Suppose $U \in \mathcal{T}$.
  - Then $\forall x \in U, \exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subseteq U$.
  - Hence $U = \bigcup_{x \in U} B_x$.
  - So $U \in \left\{ \bigcup \mathcal{U} \mid \mathcal{U} \subseteq \mathcal{B} \right\}$
Basis for a Topology

The next Prop. extracts a basis from a topology

**Proposition 84 (Topology to a basis)**

Let $X$ be a set and $\mathcal{T}$ be a topology on $X$. Let $\mathcal{C} \subset \mathcal{T}$ and suppose that for all open sets $U \in \mathcal{T}$ and all $x \in U$ there is $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$.

**Proof.**

- **Claim I:** \( \forall x \in X, \exists C \in \mathcal{C} \text{ s.t. } x \in C \)
  - Suppose $x \in X$.
  - $X$ is open and $x \in X$ so by assumption $\exists C \in \mathcal{C}$ such that $x \in C \subset X$.

**Proof of Prop. 84 (continued).**

- **Claim II:** If $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$ then there is $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.
  - Suppose $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$.
  - $\mathcal{C} \subset \mathcal{T}$ so $C_1$ and $C_2$ are open.
  - By Top. Rule (3) $C_1 \cap C_2$ is open and $x \in C_1 \cap C_2$.
  - Thus by assumption $\exists C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.
  - Thus $\mathcal{C}$ is a basis for some topology say $\mathcal{T}'$.

- **Claim III:** $\mathcal{T}' \subset \mathcal{T}$.
  - Suppose $U \in \mathcal{T}$.
  - If $x \in U$ then by assumption $\exists C \in \mathcal{C}$ s.t. $x \in C \subset U$.
  - Thus $U \in \mathcal{T}$.

- **Claim IV:** $\mathcal{T} \subset \mathcal{T}'$.
  - Suppose $U \in \mathcal{T}$.
  - Then by Prop. 83 above there is some collection of open sets $U \subset \mathcal{U}$ s.t. $U = \bigcup \mathcal{U}$.
  - But $\mathcal{C} \subset \mathcal{T}$ so by Top. Rule (2) $U = \bigcup U \in \mathcal{T}'$. \( \square \)
Proposition 85

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set $X$. The following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. $\forall x \in X$ and $\forall B \in \mathcal{B}$ with $x \in B$ there is $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Proof.

$(2) \Rightarrow (1)$
- Suppose (2).
- Let $U \in \mathcal{T}$ and let $x \in U$.
- $\mathcal{B}$ generates $\mathcal{T}$ so $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.
- Hence by (2) there is $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U$.
- Hence $U \in \mathcal{T}'$.

$(1) \Rightarrow (2)$
- Suppose $\mathcal{T}'$ finer than $\mathcal{T}$.
- Let $x \in X$ and $B \in \mathcal{B}$ with $x \in B$.
- $B \in \mathcal{T}$ so by assumption $B \in \mathcal{T}'$.
- $\mathcal{T}'$ generated by $\mathcal{B}'$ so there is $B' \in \mathcal{B}'$ with $x \in B' \subset B$. 

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Basis for a Topology

**Definition 86 (Three topologies on \( \mathbb{R} \))**

1. The **standard topology** on \( \mathbb{R} \) has basis
   \[ \mathcal{B} = \{ (a, b) \mid a, b \in \mathbb{R} \} \]

2. The **lower limit topology** on \( \mathbb{R} \) (denoted \( \mathbb{R}_\ell \)) has basis
   \[ \mathcal{B}' = \{ [a, b) \mid a, b \in \mathbb{R} \} \]

3. Let
   \[ K = \{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \} \]
   The **\( K \)-topology** on \( \mathbb{R} \) (denoted \( \mathbb{R}_K \)) has basis
   \[ \mathcal{B}'' = \{ (a, b) \mid a, b \in \mathbb{R} \} \cup \{ (a, b) - K \mid a, b \in \mathbb{R} \} \]

**Lemma 87**

The set
\[ \mathcal{B} = \{ (a, b) \mid a, b \in \mathbb{R} \} \]

is a basis for a topology on \( \mathbb{R} \).

**Proof.**

- **Claim I:** \( x \in \mathbb{R} \) implies there is \( B \in \mathcal{B} \) s.t. \( x \in B \).
  - Suppose \( x \in \mathbb{R} \).
  - Then \( x \in (x - 1, x + 1) \).
- **Claim II:** If \( B_1, B_2 \in \mathcal{B} \) and \( x \in B_1 \) and \( x \in B_2 \) then there is \( B_3 \in \mathcal{B} \) s.t. \( x \in B_3 \) and \( B_3 \subset B_1 \cap B_2 \).
  - Suppose \( a, b, c, d \in \mathbb{R} \) and \( x \in (a, b) \) and \( x \in (c, d) \).
  - Then \( a < x < b \) and \( c < x < d \).
  - Let \( B_3 = (\max\{a, c\}, \min\{b, d\}) \).
  - Then \( x \in B_3 \subset (a, b) \cap (c, d) \).
Basis for a Topology

Problem 88

Let $X$ be an arbitrary set. Give a basis $\mathcal{B}$ for the discrete topology $\mathcal{T}_d = \mathcal{P}(X)$ on $X$.

Outlines for two solutions.

1. Let $\mathcal{B}_1 = \mathcal{P}(X)$.
   - Claim: If $\mathcal{T}$ is a topology then $\mathcal{T}$ is a basis for itself.
   - Can you prove this?

2. Let $\mathcal{B}_2 = \{\{x\} \mid x \in X\}$.
   - Claim: $\mathcal{B}_2$ is a basis.
   - Claim: $\mathcal{B}_2$ generates $\mathcal{P}(X)$.

Definition 89 (Subbasis)

A subbasis for a topology on $X$ is a collection of subsets $\mathcal{S} \subset \mathcal{P}(X)$ such that $\bigcup \mathcal{S} = X$.

The basis $\mathcal{B}$ generated by the subbasis $\mathcal{S}$ is the set of all finite intersections of sets in $\mathcal{S}$.

The topology $\mathcal{T}$ generated by the subbasis $\mathcal{S}$ is the topology generated by the basis $\mathcal{B}$.

Proposition 90

If $\mathcal{S}$ is a subbasis then the set of all finite intersections of sets in $\mathcal{B}$ is a basis.
Proof of Prop 90.

- Let \( X \) be a set and \( S \) a subbasis for a topology on \( X \).
- Let \( \mathcal{C} \) be the set of all unions of all finite intersections of sets in \( S \).
- Wish to show \( \mathcal{C} \) is a basis.
- Claim I: If \( x \in X \) then there is \( C \in \mathcal{C} \) such that \( x \in C \).
  - Suppose \( x \in X \).
  - \( S \) is a subbasis so \( \bigcup S = X \).
  - Hence there is \( S \in \mathcal{C} \) s.t. \( x \in S \).
  - \( S = \bigcap \{S\} \) so \( S \in \mathcal{C} \).
- Claim II: If \( C_1, C_2 \in \mathcal{C} \) and \( x \in C_1 \) and \( x \in C_2 \) then there is \( C_3 \in \mathcal{B} \) s.t. \( x \in C_3 \) and \( C_3 \subset C_1 \cap C_2 \).
  - Suppose \( C_1, C_2 \in \mathcal{C} \) and \( x \in C_1 \) and \( x \in C_2 \).
  - Then \( C_1 = S_1 \cap \cdots \cap S_n \) and \( C_2 = S'_1 \cap \cdots \cap S'_m \).
  - Let \( C_3 = S_1 \cap \cdots \cap S_n \cap S'_1 \cap \cdots \cap S'_m \).