Math 5801
General Topology and Knot Theory

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Course Info

Reading for Monday, September 10

Review 2.12-2.15

HW 3 for Monday, September 10

- Chapter 1.9: 1, 2b, 6a
- Chapter 1.10: 1, 2a-b

Basis for a Topology

- The next Prop. extracts a basis from a topology

Proposition 84 (Topology to a basis)

Let X be a set and $\mathcal{T}$ be a topology on X. Let $\mathcal{C} \subset \mathcal{T}$ and suppose that for all open sets $U \in \mathcal{T}$ and all $x \in U$ there is $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$.

Proof.

- Claim I: $\forall x \in X, \exists C \in \mathcal{C}$ s.t. $x \in C$
  - Suppose $x \in X$.
  - X is open and $x \in X$ so by assumption $\exists C \in \mathcal{C}$ such that $x \in C \subset X$.

Basis for a Topology

Proof of Prop. 84 *(continued)*.

- Claim II: If $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$ then there is $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.
  - Suppose $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$.
  - $\mathcal{C} \subset \mathcal{T}$ so C_1 and C_2 are open.
  - By Top. Rule (3) $C_1 \cap C_2$ is open and $x \in C_1 \cap C_2$.
  - Thus by assumption $\exists C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$
- Thus $\mathcal{C}$ is a basis for some topology say $\mathcal{T}'$.
- Claim III: $\mathcal{T}' \subset \mathcal{T}$.
  - Suppose $U \in \mathcal{T}$
  - If $x \in U$ then by assumption $\exists C \in \mathcal{C}$ s.t. $x \in C \subset U$.
  - Thus $U \in \mathcal{T}$.
- Claim IV: $\mathcal{T} \subset \mathcal{T}'$.
  - Suppose $U \in \mathcal{T}'$
  - Then by Prop. **??** above there is some collection of open sets $\mathcal{U} \subset \mathcal{C}$ s.t. $U = \bigcup \mathcal{U}$.
  - But $\mathcal{C} \subset \mathcal{T}$ so by Top. Rule (2) $U = \bigcup \mathcal{U} \in \mathcal{T}$. □

Basis for a Topology

Proposition 85

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set X. The following are equivalent:

1. *$\mathcal{T}'$ is finer than $\mathcal{T}$.*
2. *$\forall x \in X$ and $\forall B \in \mathcal{B}$ with $x \in B$ there is $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

Proof.

(2) $\Rightarrow$ (1)

- Suppose (2).
- Let $U \in \mathcal{T}$ and let $x \in U$.
- $\mathcal{B}$ generates $\mathcal{T}$ so $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.
- Hence by (2) there is $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U$.
- Hence $U \in \mathcal{T}'$.

Basis for a Topology

Proof of Prop. 85 *(continued)*.

(1) $\Rightarrow$ (2)

- Suppose $\mathcal{T}'$ finer than $\mathcal{T}$.
- Let $x \in X$ and $B \in \mathcal{B}$ with $x \in B$.
- $B \in \mathcal{T}$ so by assumption $B \in \mathcal{T}'$.
- $\mathcal{T}'$ generated by $\mathcal{B}'$ so there is $B' \in \mathcal{B}'$ with $x \in B' \subset B$.

□

Basis for a Topology

Definition 86 (Three topologies on $\mathbf{R}$)

1. The **standard topology** on $\mathbf{R}$ has basis

$$\mathcal{B} = \{(a,b) \mid a,b \in \mathbf{R}\}$$

2. The **lower limit topology** on $\mathbf{R}$ (denoted $\mathbf{R}_\ell$) has basis

$$\mathcal{B}' = \{[a,b) \mid a,b \in \mathbf{R}\}$$

3. Let

$$K = \{\tfrac{1}{n} \mid n \in \mathbf{Z}_+\}$$

The K-**topology** on $\mathbf{R}$ (denoted $\mathbf{R}_K$) has basis

$$\mathcal{B}'' = \{(a,b) \mid a,b \in \mathbf{R}\} \cup \{(a,b) - K \mid a,b \in \mathbf{R}\}$$

Lecture 7 - 9/7/2012 Order Topology Product Topology Product Topology

Basis for a Topology

Lemma 87

The set

$$\mathcal{B} = \{(a,b) \mid a,b \in \mathbf{R}\}$$

is a basis for a topology on $\mathbf{R}$.

Proof.

- Claim I: $x \in \mathbf{R}$ implies there is $B \in \mathcal{B}$ s.t. $x \in B$.
  - Suppose $x \in \mathbf{R}$.
  - Then $x \in (x-1, x+1)$.
- Claim II: If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1$ and $x \in B_2$ then there is $B_3 \in \mathcal{B}$ s.t. $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.
  - Suppose $a,b,c,d \in \mathbf{R}$ and $x \in (a,b)$ and $x \in (c,d)$.
  - Then $a < x < b$ and $c < x < d$.
  - Let $B_3 = (\max\{a,c\}, \min\{b,d\})$.
  - Then $x \in B_3 \subset (a,b) \cap (c,d)$.

□

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Basis for a Topology

Problem 88

Let X be an arbitrary set. Give a basis $\mathcal{B}$ for the discrete topology $\mathcal{T}_d = \mathcal{P}(X)$ on X.

Outlines for two solutions.

1. Let $\mathcal{B}_1 = \mathcal{P}(X)$.
   - Claim: If $\mathcal{T}$ is a topology then $\mathcal{T}$ is a basis for itself.
   - Can you prove this?
2. Let $\mathcal{B}_2 = \left\{\{x\} \,\middle|\, x \in X\right\}$.
   - Claim: $\mathcal{B}_2$ is a basis.
   - Claim: $\mathcal{B}_2$ generates $\mathcal{P}(X)$.

Basis for a Topology

Definition 89 (Subbasis)

A **subbasis** for a topology on X is a collection of subsets $\mathcal{S} \subset \mathcal{P}(X)$ such that $\bigcup \mathcal{S} = X$.

The **basis** $\mathcal{B}$ **generated by the subbasis** $\mathcal{S}$ is the set of all finite intersections of sets in $\mathcal{S}$.

The **topology** $\mathcal{T}$ **generated by the subbasis** $\mathcal{S}$ is the topology generated by the basis $\mathcal{B}$.

Proposition 90

If $\mathcal{S}$ is a subbasis then the set of all finite intersections of sets in $\mathcal{B}$ is a basis.

Basis for a Topology

Proof of Prop 90.

- Let X be a set and $\mathcal{S}$ a subbasis for a topology on X.
- Let $\mathcal{C}$ be the set of all unions of all finite intersections of sets in $\mathcal{S}$
- Wish to show $\mathcal{C}$ is a basis.
- Claim I: If $x \in X$ then there is $C \in \mathcal{C}$ such that $x \in C$.
  - Suppose $x \in X$.
  - $\mathcal{S}$ is a subbasis so $\bigcup \mathcal{S} = X$.
  - Hence there is $S \in \mathcal{S}$ s.t. $x \in S$.
  - $S = \bigcap\{S\}$ so $S \in \mathcal{C}$.
- Claim II: If $C_1, C_2 \in \mathcal{C}$ and $x \in C_1$ and $x \in C_2$ then there is $C_3 \in \mathcal{B}$ s.t. $x \in C_3$ and $C_3 \subset C_1 \cap C_2$.
  - Suppose $C_1, C_2 \in \mathcal{C}$ and $x \in C_1$ and $x \in C_2$.
  - Then $C_1 = S_1 \cap \cdots \cap S_n$ and $C_2 = S_1' \cap \cdots \cap S_m'$.
  - Let $C_3 = S_1 \cap \cdots \cap S_n \cap S_1' \cap \cdots \cap S_m'$.

□

Order Topology

Definition 91 (Order Topology)

Let X be a totally ordered set. The **order topology** on X has basis

$$\begin{aligned}\mathcal{B} =& \left\{(a,b) \,\middle|\, a,b \in X\right\} \\ & \cup \left\{(a,b] \,\middle|\, a,b \in X \text{ and } b \text{ maximal in } X\right\} \\ & \cup \left\{[a,b) \,\middle|\, a,b \in X \text{ and } a \text{ minimal in } X\right\}\end{aligned}$$

Proposition 92

If X is totally ordered then the set $\mathcal{B}$ above is a basis.

- For proof slightly modify proof of Lemma 87 above.
- **R** with order topology is exactly **R** with standard topology.
- What is **Z** with order topology?

Product Topology

Definition 93 (Product Topology)

Let X and Y be topological spaces. The **product topology** on $X \times Y$ has basis

$$\mathcal{B} = \left\{ U \times V \,\middle|\, U \text{ open in } X \text{ and } V \text{ open in } Y \right\}$$

Proposition 94

If X and Y are topological spaces then the set $\mathcal{B}$ above is a basis.

Product Topology

Proof of Prop. 94.

- Claim I: $(x, y) \in X \times Y$ implies there is $B \in \mathcal{B}$ s.t. $(x, y) \in B$.
  - Let $B = X \times Y$.
- Claim II: If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1$ and $x \in B_2$ then there is $B_3 \in \mathcal{B}$ s.t. $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.
  - Let $(x, y) \in U_1 \times V_1$ and $(x, y) \in U_2 \times V_2$.
  - Let $B_3 = (U_1 \cap U_2) \times (V_1 \cap V_2)$.

□