Course Info

Reading for Friday, September 14
Chapter 2.18, pgs. 102-111

HW 4 for Monday, September 17
- Chapter 2.13: 1, 3, 5, 8a
- Chapter 2.16: 1, 4, 6, 9
Closed Sets

Proposition 113 (De Morgan’s Laws)

If \( X \) is a set \( A \subset X \) and \( B \subset X \) and \( S \subset \mathcal{P}(X) \) is nonempty then

1. \( X - (A \cup B) = (X - A) \cap (X - B) \).
2. \( X - (A \cap B) = (X - A) \cup (X - B) \).
3. \( X - \bigcup S = \bigcap \{ X - S | S \in S \} \).
4. \( X - \bigcap S = \bigcup \{ X - S | S \in S \} \).

Proof.

▶ (1) and (2) are special cases of (3) and (4) resp.
▶ Proof of (3) (proof of (4) similar):

\[
X - \bigcup S = \{ x \in X | x \notin \bigcup S \} \\
= \{ x \in X | \forall S \in S, x \notin S \} \\
= \{ x \in X | \forall S \in S, x \in X - S \} \\
= \bigcap \{ X - S | S \in S \} \\
\]

Definition 114 (Neighborhood)

If \( X \) is a space a neighborhood of \( x \) is a set \( A \) such that there is an open set \( U \) with \( x \in U \subset A \).

▶ Note book def. insists that neighborhoods be open sets.
▶ book “neighborhood” = lecture “open neighborhood”

Examples 115 (Neighborhoods)

1. In \( \mathbb{R} \) \( \text{Bd}(1, 3) \) is not a neighborhood of 3.
2. In \( \mathbb{R} \) \( [2, \pi] \) is a neighborhood of 3.
3. In \( X \forall x \in X \) \( X \) is a nbhd. of \( x \).

Proposition 116

Let \( X \) be a space with basis \( \mathcal{B} \). Then \( \forall x \in X \) every neighborhood \( N_x \) of \( x \) contains a basis element \( B_x \in \mathcal{B} \) with \( x \in B_x \).
Closed Sets

Lemma 117 (Neighborhood Criterion for Open/Closed Sets)

Let $X$ be a space.

1. A subset $U \subset X$ is open iff $\forall x \in U, \exists$ a nbhd $N_x$ for $x$ with $N_x \subset U$.
2. A subset $C \subset X$ is closed iff $\forall x \in X - C, \exists$ a nbhd $N_x$ for $x$ with $N_x \subset X - C$.

Proof.

- Note that (2) follows from (1) and def of closed set.
- Claim I: $U \subset X$ is open $\Rightarrow \forall x \in U, \exists$ a nbhd $N_x$ for $x$ with $N_x \subset U$
  - Suppose $U \subset X$ is open and $x \in U$
  - Then $U$ is a neighborhood of $x$ and $U \subset U$ so let $N_x = U$.
  - Now suppose $U \subset X$ and $\forall x \in U, \exists$ a nbhd $N_x$ for $x$ with $N_x \subset U$.
- Claim II: $\forall x \in U, \exists$ a nbhd $N_x$ for $x$ with $N_x \subset U \Rightarrow U$ open
  - Suppose $U \subset X$ and $\forall x \in U, \exists$ a nbhd $N_x$ for $x$ with $N_x \subset U \Rightarrow U$
  - For each $x$, $N_x$ contains an open set $U_x$ with $x \in U_x \subset N_x \subset U$
  - $U = \bigcup_{x \in U} U_x$ so $U$ is a union of open sets.

Lemma 118 (Neighborhood Criterion for Open/Closed Sets)

Let $X$ be a space and $A \subset X$.

1. $\overline{A} = X - \text{Int}(X - A)$
2. $\text{Int} A = X - \overline{X - A}$
3. $x \in \text{Int} A$ iff $x$ has a nbhd $N_x$ s.t. $x \in N_x \subset A$
4. $x \notin \overline{A}$ iff $x$ has a nbhd $N_x$ s.t. $x \in N_x \subset X - A$

Proof.

- Proof of (1) (proof of (2) similar):
  - $\overline{A} = \bigcap \{C | C$ is a closed and $A \subset C\}$
  - $= \bigcap \{X - U | U$ is open and $A \subset (X - U)\}$
  - $= X - \bigcup \{U | U$ is open and $A \subset (X - U)\}$
  - $= X - \bigcup \{U | U$ is open and $A \subset (X - A)\} = X - \text{Int}(X - A)$
Closed Sets

Definition 119
Let $A$ and $B$ be sets. We say $A$ intersects $B$ if $A \cap B \neq \emptyset$.

Proposition 120
Let $X$ be a space and $A \subset X$. Let $\mathcal{B}$ be a basis for $X$. TFAE

1. $x \in \overline{A}$
2. Every neighborhood $N_x$ of $x$ intersects $A$.
3. Every open neighborhood $N_x$ of $x$ intersects $A$.
4. Every neighborhood $N_x \in \mathcal{B}$ of $x$ intersects $A$.

Proof.
$x \in \overline{A} \iff x$ has no nbhd $N_x$ s.t. $N_x \subset X - A$
$\iff$ every nbhd $N_x$ of $x$ intersects $A$
$\iff$ every open nbhd $N_x$ of $x$ intersects $A$
$\iff$ every nbhd $N_x \in \mathcal{B}$ of $x$ intersects $A$

Definition 121 (Hausdorff)
A space $X$ is Hausdorff if for all $x, y \in X$ with $x \neq y$ there are neighborhoods $N_x$ and $N_y$ of $x$ and $y$ resp. such that $N_x$ and $N_y$ are disjoint.

Definition 122 ($T_1$ Axiom)
A space $X$ satisfies the $T_1$ Axiom if for all $x \in X$ the set $\{x\}$ is closed.
Closed Sets

**Proposition 123**

*If $X$ is Hausdorff then $X$ satisfies the $T_1$ Axiom*

**Proof.**

- Suppose $X$ Hasudorff and $x, y \in X$ and $x \neq y$.
- There is a nbhd $N_y$ of $y$ which does not intersect $\{x\}$.
- Thus $y \notin \{x\}$.
- Hence $\{x\} = \{x\}$.
- Hence every one-point set is closed.

**Example 124 ($T_1$ Axiom strictly weaker than Hausdorff)**

$\mathbb{Z}_f$ satisfies the $T_1$ axiom but is not Hausdorff.

Closed Sets

**Proposition 125**

*If $X$ satisfies the $T_1$ Axiom then finite sets are closed.*

**Proof.**

- If $X$ is $T_1$ each $\{x\}$ is closed.
- Finite unions of closed sets are closed.
- Thus finite sets are closed.
Definition 126 (Convergent Sequence)

Let $X$ be a space. A sequence $(x_n)_{n \in \mathbb{Z}^+}$ in $X$ converges to $x \in X$ if for every open nbhd $U_x$ of $x$ there is $N \in \mathbb{Z}^+$ s.t. for all $n > N$ we have $x_n \in U_x$.

Examples 127 (Convergent Sequences)

1. In $\mathbb{R}$ we have $\frac{1}{n}$ converges to 0.
2. In $\mathbb{R}_f$ we have $\frac{1}{n}$ converges to $\pi$.
3. In $\mathbb{R}_f$ we have $(-1)^n$ does not converge.
4. In $\mathbb{R}_f$ we have $(1)^n$ converges to 1.

Proposition 128 (Limits Unique in Hausdorff Spaces)

Let $X$ be a Hausdorff space and let $(x_n)_{n \in \mathbb{Z}^+}$ be a convergent sequence in $X$. Then there is a unique $x \in X$ s.t. $(x_n)_{n \in \mathbb{Z}^+}$ converges to $x$.

Proof.

- Suppose $x, y \in X$ with $x \neq y$ and $(x_n)_{n \in \mathbb{Z}^+}$ converges to both $x$ and $y$.
- Then we have disjoint nbhds $N_x$ and $N_y$ for $x$ and $y$.
- Thus there are $M, L \in \mathbb{Z}^+$ s.t. for all $n > M$ $x_n \in N_x$ and for all $n > L$ $x_n \in N_y$ and
- Choose $m$ bigger than $M$ and $L$.
- Then $x_m \in N_x$ and $x_m \in N_y$ contradicting disjointness.