

#### Nathan Broaddus

Ohio State University

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	Lecture 10 - 9/14/2012	Closed Sets
	Lecture 10 - 9/14/2012	CIURD DEC
Course Info		

- Chapter 2.13. 1, 5, 5, 6a
- Chapter 2.16: 1, 4, 6, 9

## Proposition 113 (De Morgan's Laws)

If X is a set  $A \subset X$  and  $B \subset X$  and  $S \subset \mathcal{P}(X)$  is nonempty then

- 1.  $X (A \cup B) = (X A) \cap (X B)$ .
- 2.  $X (A \cap B) = (X A) \cup (X B)$ .
- 3.  $X \bigcup S = \bigcap \{X S | S \in S\}.$
- 4.  $X \bigcap S = \bigcup \{X S | S \in S\}.$

## Proof.

- (1) and (2) are special cases of (3) and (4) resp.
- Proof of (3) (proof of (4) similar):

$$\begin{aligned} X - \bigcup \$ &= \{x \in X | x \notin \bigcup \$\} \\ &= \{x \in X | \forall S \in \$, x \notin S\} \\ &= \{x \in X | \forall S \in \$, x \in X - S\} \\ &= \bigcap \{X - S | S \in \$\} \end{aligned}$$
We the Bracket

## **Closed Sets**

## Definition 114 (Neighborhood)

If X is a space a **neighborhood** of x is a set A such that there is an open set U with  $x \in U \subset A$ .

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- Note book def. insists that neighborhoods be open sets.
- book "neighborhood" = lecture "open neighborhood"

## Examples 115 (Neighborhoods)

- 1. In R Bd(1,3] is not a neighborhood of 3.
- 2. In **R**  $[2, \pi]$  is a neighborhood of 3.
- 3. In  $X \forall x \in X X$  is a nbhd. of x.

## Proposition 116

Let X be a space with basis  $\mathfrak{B}$ . Then  $\forall x \in X$  every neighborhood  $N_x$  of x contains a basis element  $B_x \in \mathfrak{B}$  with  $x \in \mathfrak{B}$ .

Lemma 117 (Neighborhood Criterion for Open/Closed Sets)

Let X be a space.

- 1. A subset  $U \subset X$  is open iff  $\forall x \in U, \exists$  a nbhd  $N_x$  for x with  $N_x \subset U$ .
- 2. A subset  $C \subset X$  is closed iff  $\forall x \in X C, \exists$  a nbhd  $N_x$  for x with  $N_x \subset X C$ .

## Proof.

- Note that (2) follows from (1) and def of closed set.
- Claim I: U ⊂ X is open ⇒ ∀x ∈ U, ∃ a nbhd N<sub>x</sub> for x with N<sub>x</sub> ⊂ U
  - Suppose U ⊂ X is open and x ∈ U
  - Then U is a neighborhood of x and U ⊂ U so let N<sub>x</sub> = U.
  - Now suppose U ⊂ X and ∀x ∈ U, ∃ a nbhd N<sub>x</sub> for x with N<sub>x</sub> ⊂ U.
- Claim II: ∀x ∈ U, ∃ a nbhd N<sub>x</sub> for x with N<sub>x</sub> ⊂ U ⇒ U open
  - Suppose U ⊂ X and ∀x ∈ U,∃ a nbhd N<sub>x</sub> for x with N<sub>x</sub> ⊂ U ⇒ U
  - For each x, N<sub>x</sub> contains an open set U<sub>x</sub> with x ∈ U<sub>x</sub> ⊂ N<sub>x</sub> ⊂ U
  - ►  $U = \bigcup_{x \in U} U_x$  so U is a union of open sets. Nathan Broaddus General Topology and Knot Theory

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Lemma 118 (Neighborhood Criterion for Open/Closed Sets)

Let X be a space and  $A \subset X$ .

- 1.  $\overline{A} = X \operatorname{Int}(X A)$
- 2. Int  $A = X \overline{X A}$
- 3.  $x \in Int A$  iff x has a nbhd  $N_x$  s.t.  $x \in N_x \subset A$
- 4.  $x \notin \overline{A}$  iff x has a nbhd  $N_x$  s.t.  $x \in N_x \subset X A$

## Proof.

Proof of (1) (proof of (2) similar):

$$\overline{A} = \bigcap \{ C | C \text{ is a closed and } A \subset C \}$$

- $= \bigcap \{X U | U \text{ is open and } A \subset (X U)\}$
- $= X \bigcup \{ U | U \text{ is open and } A \subset (X U) \}$
- $= X [] \{ U | U \text{ is open and } U \subset (X A) \} = X Int(X A)$

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# Closed Sets

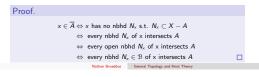
#### Definition 119

Let A and B be sets. We say A intersects B if  $A \cap B \neq \emptyset$ .

#### Proposition 120

Let X be a space and  $A \subset X$ . Let B be a basis for X. TFAE

- 1.  $x \in \overline{A}$
- 2. Every neighborhood N<sub>x</sub> of x intersects A.
- 3. Every open neighborhood N<sub>x</sub> of x intersects A.
- 4. Every neighborhood  $N_x \in \mathcal{B}$  of x intersects A.



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## Definition 121 (Hausdorff)

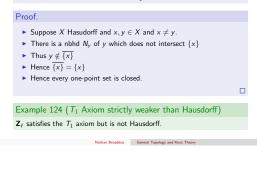
A space X is **Hausdorff** if for all  $x, y \in X$  with  $x \neq y$  there are neighborhoods  $N_x$  and  $N_y$  of x and y resp. such that  $N_x$  and  $N_y$  are disjoint.

Definition 122 ( $T_1$  Axiom)

A space X satisfies the T<sub>1</sub> Axiom if for all  $x \in X$  the set  $\{x\}$  is closed.

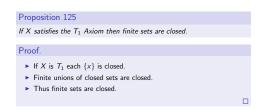
### **Proposition 123**

If X is Hausdorff then X satisfies the T1 Axiom



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#### Definition 126 (Convergent Sequence)

Let X be a space. A sequence  $(x_n)_{n \in \mathbb{Z}_+}$  in X converges to  $x \in X$  if for every open nbhd  $U_x$  of x there is  $N \in \mathbb{Z}_+$  s.t. for all n > N we have  $x_n \in U_x$ .

## Examples 127 (Convergent Sequences)

- 1. In **R** we have  $\frac{1}{n}$  coverges to 0.
- 2. In  $\mathbf{R}_f$  we have  $\frac{1}{n}$  converges to  $\pi$ .
- 3. In  $\mathbf{R}_f$  we have  $(-1)^n$  does not converge.
- In R<sub>f</sub> we have (1)<sup>n</sup> converges to 1.

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## Proposition 128 (Limits Unique in Hausdorff Spaces)

Let X be a Hausdorff space and let  $(x_n)_{n \in \mathbb{Z}_+}$  be a convergent sequence in X. Then there is a unique  $x \in X$  s.t.  $(x_n)_{n \in \mathbb{Z}_+}$  converges to x.

## Proof.

- Suppose  $x, y \in X$  with  $x \neq y$  and  $(x_n)_{n \in \mathbb{Z}_+}$  converges to both x and y.
- Then we have disjoint nbhds N<sub>x</sub> and N<sub>y</sub> for x and y.
- ▶ Thus there are  $M, L \in \mathbb{Z}_+$  s.t. for all  $n > M x_n \in N_x$  and for all  $n > L x_n \in N_y$  and
- Choose m bigger than M and L.
- ► Then x<sub>m</sub> ∈ N<sub>x</sub> and x<sub>m</sub> ∈ N<sub>y</sub> contradicting disjointness.