# Math 5801
# General Topology and Knot Theory

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## Course Info

### Reading for Friday, September 14

Chapter 2.18, pgs. 102-111

### HW 4 for Monday, September 17

- Chapter 2.13: 1, 3, 5, 8a
- Chapter 2.16: 1, 4, 6, 9

## Closed Sets

### Proposition 113 (De Morgan's Laws)

*If $X$ is a set $A \subset X$ and $B \subset X$ and $\mathcal{S} \subset \mathcal{P}(X)$ is nonempty then*

1. $X - (A \cup B) = (X - A) \cap (X - B)$.
2. $X - (A \cap B) = (X - A) \cup (X - B)$.
3. $X - \bigcup \mathcal{S} = \bigcap \{X - S | S \in \mathcal{S}\}$.
4. $X - \bigcap \mathcal{S} = \bigcup \{X - S | S \in \mathcal{S}\}$.

### Proof.

- (1) and (2) are special cases of (3) and (4) resp.
- Proof of (3) (proof of (4) similar):
  - $$\begin{aligned} X - \bigcup \mathcal{S} &= \{x \in X | x \notin \bigcup \mathcal{S}\} \\ &= \{x \in X | \forall S \in \mathcal{S},\ x \notin S\} \\ &= \{x \in X | \forall S \in \mathcal{S},\ x \in X - S\} \\ &= \bigcap \{X - S | S \in \mathcal{S}\} \end{aligned}$$

□





## Closed Sets

### Definition 114 (Neighborhood)

If $X$ is a space a **neighborhood** of $x$ is a set $A$ such that there is an open set $U$ with $x \in U \subset A$.

- Note book def. insists that neighborhoods be open sets.
- book "neighborhood" = lecture "open neighborhood"

### Examples 115 (Neighborhoods)

1. In $\mathbf{R}$ Bd$(1, 3]$ is not a neighborhood of 3.
2. In $\mathbf{R}$ $[2, \pi]$ is a neighborhood of 3.
3. In $X$ $\forall x \in X$ $X$ is a nbhd. of $x$.

### Proposition 116

*Let $X$ be a space with basis $\mathcal{B}$. Then $\forall x \in X$ every neighborhood $N_x$ of $x$ contains a basis element $B_x \in \mathcal{B}$ with $x \in \mathcal{B}$.*

# Closed Sets

## Lemma 117 (Neighborhood Criterion for Open/Closed Sets)

*Let $X$ be a space.*

1. *A subset $U \subset X$ is open iff $\forall x \in U, \exists$ a nbhd $N_x$ for $x$ with $N_x \subset U$.*
2. *A subset $C \subset X$ is closed iff $\forall x \in X - C, \exists$ a nbhd $N_x$ for $x$ with $N_x \subset X - C$.*

## Proof.

- Note that (2) follows from (1) and def of closed set.
- Claim I: $U \subset X$ is open $\Rightarrow \forall x \in U, \exists$ a nbhd $N_x$ for $x$ with $N_x \subset U$
  - Suppose $U \subset X$ is open and $x \in U$
  - Then $U$ is a neighborhood of $x$ and $U \subset U$ so let $N_x = U$.
  - Now suppose $U \subset X$ and $\forall x \in U, \exists$ a nbhd $N_x$ for $x$ with $N_x \subset U$.
- Claim II: $\forall x \in U, \exists$ a nbhd $N_x$ for $x$ with $N_x \subset U \Rightarrow U$ open
  - Suppose $U \subset X$ and $\forall x \in U, \exists$ a nbhd $N_x$ for $x$ with $N_x \subset U \Rightarrow U$
  - For each $x$, $N_x$ contains an open set $U_x$ with $x \in U_x \subset N_x \subset U$
  - $U = \bigcup_{x \in U} U_x$ so $U$ is a union of open sets. □





# Closed Sets

## Lemma 118 (Neighborhood Criterion for Open/Closed Sets)

*Let $X$ be a space and $A \subset X$.*

1. $\overline{A} = X - \text{Int}(X - A)$
2. $\text{Int}\, A = X - \overline{X - A}$
3. $x \in \text{Int}\, A$ *iff $x$ has a nbhd $N_x$ s.t. $x \in N_x \subset A$*
4. $x \notin \overline{A}$ *iff $x$ has a nbhd $N_x$ s.t. $x \in N_x \subset X - A$*

## Proof.

- Proof of (1) (proof of (2) similar):
  - $$\begin{aligned}\overline{A} &= \bigcap\{C | C \text{ is a closed and } A \subset C\}\\ &= \bigcap\{X - U | U \text{ is open and } A \subset (X - U)\}\\ &= X - \bigcup\{U | U \text{ is open and } A \subset (X - U)\}\\ &= X - \bigcup\{U | U \text{ is open and } U \subset (X - A)\} = X - \text{Int}(X - A)\end{aligned}$$

# Closed Sets

## Definition 119

Let $A$ and $B$ be sets. We say $A$ **intersects** $B$ if $A \cap B \neq \varnothing$.

## Proposition 120

*Let $X$ be a space and $A \subset X$. Let $\mathcal{B}$ be a basis for $X$. TFAE*

1. $x \in \overline{A}$
2. *Every neighborhood $N_x$ of $x$ intersects $A$.*
3. *Every open neighborhood $N_x$ of $x$ intersects $A$.*
4. *Every neighborhood $N_x \in \mathcal{B}$ of $x$ intersects $A$.*

## Proof.

$$\begin{aligned} x \in \overline{A} &\Leftrightarrow x \text{ has no nbhd } N_x \text{ s.t. } N_x \subset X - A \\ &\Leftrightarrow \text{ every nbhd } N_x \text{ of x intersects } A \\ &\Leftrightarrow \text{ every open nbhd } N_x \text{ of x intersects } A \\ &\Leftrightarrow \text{ every nbhd } N_x \in \mathcal{B} \text{ of x intersects } A \end{aligned}$$

□





# Closed Sets

## Definition 121 (Hausdorff)

A space $X$ is **Hausdorff** if for all $x, y \in X$ with $x \neq y$ there are neighborhoods $N_x$ and $N_y$ of $x$ and $y$ resp. such that $N_x$ and $N_y$ are disjoint.

## Definition 122 ($T_1$ Axiom)

A space $X$ satisfies the $\mathbf{T_1}$ **Axiom** if for all $x \in X$ the set $\{x\}$ is closed.

# Closed Sets

## Proposition 123

*If $X$ is Hausdorff then $X$ satisfies the $T_1$ Axiom*

## Proof.

- Suppose $X$ Hasudorff and $x, y \in X$ and $x \neq y$.
- There is a nbhd $N_y$ of $y$ which does not intersect $\{x\}$
- Thus $y \notin \overline{\{x\}}$
- Hence $\overline{\{x\}} = \{x\}$
- Hence every one-point set is closed.

□

## Example 124 ($T_1$ Axiom strictly weaker than Hausdorff)

$\mathbf{Z}_f$ satisfies the $T_1$ axiom but is not Hausdorff.





# Closed Sets

## Proposition 125

*If $X$ satisfies the $T_1$ Axiom then finite sets are closed.*

## Proof.

- If $X$ is $T_1$ each $\{x\}$ is closed.
- Finite unions of closed sets are closed.
- Thus finite sets are closed.

□

# Closed Sets

## Definition 126 (Convergent Sequence)

Let $X$ be a space. A sequence $(x_n)_{n\in\mathbf{Z}_+}$ in $X$ **converges to** $x \in X$ if for every open nbhd $U_x$ of $x$ there is $N \in \mathbf{Z}_+$ s.t. for all $n > N$ we have $x_n \in U_x$.

## Examples 127 (Convergent Sequences)

1. In $\mathbf{R}$ we have $\frac{1}{n}$ coverges to 0.
2. In $\mathbf{R}_f$ we have $\frac{1}{n}$ converges to $\pi$.
3. In $\mathbf{R}_f$ we have $(-1)^n$ does not converge.
4. In $\mathbf{R}_f$ we have $(1)^n$ converges to 1.





# Closed Sets

## Proposition 128 (Limits Unique in Hausdorff Spaces)

*Let $X$ be a Hausdorff space and let $(x_n)_{n\in\mathbf{Z}_+}$ be a convergent sequence in $X$. Then there is a unique $x \in X$ s.t. $(x_n)_{n\in\mathbf{Z}_+}$ converges to $x$.*

## Proof.

- Suppose $x, y \in X$ with $x \neq y$ and $(x_n)_{n\in\mathbf{Z}_+}$ converges to both $x$ and $y$.
- Then we have disjoint nbhds $N_x$ and $N_y$ for $x$ and $y$.
- Thus there are $M, L \in \mathbf{Z}_+$ s.t. for all $n > M$ $x_n \in N_x$ and for all $n > L$ $x_n \in N_y$ and
- Choose $m$ bigger than $M$ and $L$.
- Then $x_m \in N_x$ and $x_m \in N_y$ contradicting disjointness.

□