Math 5801
General Topology and Knot Theory

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Course Info

Reading for Wednesday, September 19

Chapter 2.19, pgs. 112-118

HW 5 for Monday, September 24

- Chapter 2.17: 3, 5, 9, 13
- Chapter 2.18: 2, 5, 8a-b, 10

Midterm 1 Friday, September 28

- Munkres Chapters 1.1-2.19
- ZFC proofs (I'll supply you with all of the axioms)

Continuous Functions

Definition 113 (Continuous Function)

Let X and Y be topological spaces. A function $f : X \to Y$ is **continuous** if for every open set $V \subset Y$ we have $f^{-1}(V)$ is open in X.

A continuous function is also called a **map**.

Definition 114 ((δ, ϵ)-continuity)

A function $f : \mathbf{R} \to \mathbf{R}$ is (δ, ε)-**continuous** if for all $a \in \mathbf{R}$ and all $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in \mathbf{R}$ if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

Proposition 115 (Continuity generalizes (δ, ϵ)-continuity)

*A function $f : \mathbf{R} \to \mathbf{R}$ is (δ, ε)-**continuous** iff it is continuous.*

Continuous Functions

Proof of Prop. 115.

- Claim I: f cont. $\Rightarrow f$ is δ, ε-cont.
  - Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous and let $a \in \mathbf{R}$ and $\varepsilon > 0$.
  - Consider the open interval $V = (f(a) - \varepsilon, f(a) + \varepsilon)$.
  - f is cont. so $f^{-1}(V)$ is open.
  - $f(a) \in V$ so $a \in f^{-1}(V)$.
  - There must be a basis element (c, d) for $\mathbf{R}$ with $a \in (c, d) \subset f^{-1}(V)$.
  - Let $\delta = \min\{|a - c|, |a - d|\}$.
  - Suppose $|x - a| < \delta$.
  - Then $x \in (a - \delta, a + \delta) \subset (c, d) \subset f^{-1}(V)$
  - Hence $f(x) \in V = (f(a) - \varepsilon, f(a) + \varepsilon)$
  - If follows that $|f(x) - f(a)| < \varepsilon$.

Continuous Functions

Proof of Prop. 115 *(continued)*.

- Claim II: f is (δ, ε)-cont. $\Rightarrow f$ cont.
  - Suppose $f : \mathbf{R} \to \mathbf{R}$ is (δ, ε)-cont. and suppose $V \subset \mathbf{R}$ is open.
  - If $f^{-1}(V) = \varnothing$ then we are done. Assume $a \in f^{-1}(V)$.
  - Then $f(a) \in V$ so there is a basis elt. $(c, d) \subset \mathbf{R}$ with $f(a) \in (c, d) \subset V$.
  - Let $\varepsilon = \min\{|f(a) - c|, |f(a) - d|\}$.
  - Then there is $\delta > 0$ s.t. $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$
  - Hence $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon) \subset (c, d) \subset V$.
  - Thus $(a - \delta, a + \delta) \subset f^{-1}(V)$.
  - We see thaty every element $a \in f^{-1}(V)$ has a nbhd $(a - \delta, a + \delta) \subset f^{-1}(V)$.
  - Thus $f^{-1}(V)$ is open.
- Therefore continuity generalizes (δ, ε)-continuity.

□

Continuous Functions

Examples 116

1. If X_d is has the discrete topology then any function $f : X_d \to Y$ is continuous.
2. If Y_t has the trivial topology then any function $f : X \to Y_t$ is continuous.
3. For any space X the identity function $\mathrm{id}_X : X \to X$ is continuous.
4. $f : \mathbf{R} \to \mathbf{R}$ with $f(x) = x^2$ is (δ, ε)-continuous and hence continuous.
5. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X with $\mathcal{T}$ finer than $\mathcal{T}'$ (that is $\mathcal{T}' \subset \mathcal{T}$). Then if $f : X \to Y$ is continuous under topology $\mathcal{T}'$ then it is continuous under topology $\mathcal{T}$.
6. Let $\mathcal{S}$ and $\mathcal{S}'$ be two topologies on Y with $\mathcal{S}$ coarser than $\mathcal{S}'$ (that is $\mathcal{S} \subset \mathcal{S}'$). Then if $f : X \to Y$ is continuous under topology $\mathcal{S}'$ then it is continuous under topology $\mathcal{S}$.

Continuous Functions

Definition 117 (Continuity at a point)

Let X and Y be topological spaces. A function $f : X \to Y$ is **continuous at** $x \in X$ if for every open neighborhood V of $f(x)$ there is an open neighborhood U of x such that $f(U) \subset V$.

Proposition 118

Let X and Y be spaces and $f : X \to Y$ be a function. TFAE

1. *$f : X \to Y$ is continuous.*
2. *For all subsets $A \subset X$ we have $f(\overline{A}) \subset \overline{f(A)}$.*
3. *For all closed sets $B \subset Y$ the set $f^{-1}(B)$ is closed in X.*
4. *$f : X \to Y$ is continuous at x for all $x \in X$.*

Continuous Functions

Proof of Prop. 118.

- Claim I: (1) $\Rightarrow$ (2)
  - Suppose $f : X \to Y$ is continuous and let $A \subset X$.
  - Suppose $y \in f(\overline{A})$.
  - Then there is $x \in \overline{A}$ with $f(x) = y$.
  - Let V be an open neighborhood of $f(x)$.
  - Then $f^{-1}(V)$ is an open neighborhood of x so it must intersect A.
  - Let $x' \in A \cap f^{-1}(V)$
  - $f(x') \in f(A)$ and $f(x') \in V$
  - $f(x') \in f(A) \cap V$
  - Thus every nbdh V of $f(x)$ intersects $f(A)$.
  - Hence $y = f(x) \in \overline{f(A)}$

Continuous Functions

Proof of Prop. 118 *(continued)*.

- Claim II: (2) $\Rightarrow$ (3)
  - Suppose $f : X \to Y$ is a function and for every subset $A \subset X$ we have $f(\overline{A}) \subset \overline{f(A)}$.
  - Let $B \subset Y$ be a closed set.
  - Let $A = f^{-1}(B)$.
  - $f(A) = f(f^{-1}(B)) \subset B$.
  - If $x \in \overline{A}$ then
    $$f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{B} = B$$
  - Hence $x \in f^{-1}(B) = A$.
  - Thus $\overline{A} \subset A$
  - Hence $A = \overline{A}$ must be a closed set.
  - Thus $f^{-1}(B) = A$ is closed.

Continuous Functions

Proof of Prop. 118 *(continued)*.

- Claim III: (3) $\Rightarrow$ (1)
  - Suppose $f : X \to Y$ is a function and for every closed set $B \subset Y$ the set $f^{-1}(B)$ is closed.
  - Let $V \subset Y$ be open.
  - Let $B = Y - V$ so B is closed.
  - $$f^{-1}(B) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$$
  - By assumption $f^{-1}(B)$ is closed so $X - f^{-1}(V)$ is closed.
  - Hence $f^{-1}(V)$ is open.

Continuous Functions

Proof of Prop. 118 *(continued).*

- Claim IV: (1) $\Rightarrow$ (4)
  - Suppose $f : X \to Y$ is continuous and $x \in X$.
  - Then $U = f^{-1}(V)$ is an open neighborhood of x with $f(U) \subset V$.
  - Thus f is continuous at x for all $x \in X$.
- Claim V: (4) $\Rightarrow$ (1)
  - Suppose $f : X \to Y$ is continuous at x for all $x \in X$.
  - Let $V \subset Y$ be open.
  - If $f^{-1}(V)$ is empty we are done so let $x \in f^{-1}(V)$.
  - Then $f(x) \in V$.
  - f is continous at X so there is a nbhd U_x of x s.t. $f(U_x) \subset V$.
  - Hence $U_x \subset f^{-1}(V)$
  - Thus $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$
  - Hence $f^{-1}(V)$ is open.
  - It follows that f is continuous.

□

Continuous Functions

Definition 119 (Homeomorphism)

Let X and Y be topological spaces. A function $f : X \to Y$ is a **homeomorphism** if

- $f : X \to Y$ is a bijection
- $f : X \to Y$ is a continuous.
- $f^{-1} : Y \to X$ is continuous.