Course Info

Reading for Wednesday, September 19
Chapter 2.19, pgs. 112-118

HW 5 for Monday, September 24
- Chapter 2.17: 3, 5, 9, 13
- Chapter 2.18: 2, 5, 8a-b, 10

Midterm 1 Friday, September 28
- Munkres Chapters 1.1-2.19
- ZFC proofs (I'll supply you with all of the axioms)
Continuous Functions

Definition 113 (Continuous Function)

Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is **continuous** if for every open set $V \subset Y$ we have $f^{-1}(V)$ is open in $X$.

A continuous function is also called a **map**.

Definition 114 ($(\delta, \epsilon)$-continuity)

A function $f : \mathbb{R} \to \mathbb{R}$ is **$(\delta, \epsilon)$-continuous** if for all $a \in \mathbb{R}$ and all $\epsilon > 0$ there is $\delta > 0$ such that for all $x \in \mathbb{R}$ if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

Proposition 115 (Continuity generalizes $(\delta, \epsilon)$-continuity)

A function $f : \mathbb{R} \to \mathbb{R}$ is **$(\delta, \epsilon)$-continuous** iff it is continuous.

Proof of Prop. 115.

- **Claim I:** $f$ cont. $\Rightarrow$ $f$ is $(\delta, \epsilon)$-cont.
  - Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and let $a \in \mathbb{R}$ and $\epsilon > 0$.
  - Consider the open interval $V = (f(a) - \epsilon, f(a) + \epsilon)$.
  - $f$ is cont. so $f^{-1}(V)$ is open.
  - $f(a) \in V$ so $a \in f^{-1}(V)$.
  - There must be a basis element $(c, d)$ for $\mathbb{R}$ with $a \in (c, d) \subset f^{-1}(V)$.
  - Let $\delta = \min\{|a - c|, |a - d|\}$.
  - Suppose $|x - a| < \delta$.
  - Then $x \in (a - \delta, a + \delta) \subset (c, d) \subset f^{-1}(V)$.
  - Hence $f(x) \in V = (f(a) - \epsilon, f(a) + \epsilon)$.
  - It follows that $|f(x) - f(a)| < \epsilon$.
Continuous Functions

Proof of Prop. 115 (continued).

- Claim II: \( f \) is \((\delta, \varepsilon)\)-cont. \( \Rightarrow \) \( f \) cont.
  - Suppose \( f : \mathbb{R} \rightarrow \mathbb{R} \) is \((\delta, \varepsilon)\)-cont. and suppose \( V \subset \mathbb{R} \) is open.
  - If \( f^{-1}(V) = \emptyset \) then we are done. Assume \( a \in f^{-1}(V) \).
  - Then \( f(a) \in V \) so there is a basis elt. \((c, d) \subset \mathbb{R} \) with \( f(a) \in (c, d) \subset V \).
  - Let \( \varepsilon = \min\{|f(a) - c|, |f(a) - d|\} \).
  - Then there is \( \delta > 0 \) s.t. \( |x - a| < \delta \) implies \( |f(x) - f(a)| < \varepsilon \).
  - Hence \( f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon) \subset (c, d) \subset V \).
  - Thus \( (a - \delta, a + \delta) \subset f^{-1}(V) \).
  - We see that every element \( a \in f^{-1}(V) \) has a nbhd \((a - \delta, a + \delta) \subset f^{-1}(V) \).
  - Thus \( f^{-1}(V) \) is open.
- Therefore continuity generalizes \((\delta, \varepsilon)\)-continuity.

Examples 116

1. If \( X_d \) is has the discrete topology then any function \( f : X_d \rightarrow Y \) is continuous.
2. If \( Y_t \) has the trivial topology then any function \( f : X \rightarrow Y_t \) is continuous.
3. For any space \( X \) the identity function \( \text{id}_X : X \rightarrow X \) is continuous.
4. \( f : \mathbb{R} \rightarrow \mathbb{R} \) with \( f(x) = x^2 \) is \((\delta, \varepsilon)\)-continuous and hence continuous.
5. Let \( \mathcal{T} \) and \( \mathcal{T}' \) be two topologies on \( X \) with \( \mathcal{T} \) finer than \( \mathcal{T}' \) (that is \( \mathcal{T}' \subset \mathcal{T} \)). Then if \( f : X \rightarrow Y \) is continuous under topology \( \mathcal{T}' \) then it is continuous under topology \( \mathcal{T} \).
6. Let \( S \) and \( S' \) be two topologies on \( Y \) with \( S \) coarser than \( S' \) (that is \( S \subset S' \)). Then if \( f : X \rightarrow Y \) is continuous under topology \( S' \) then it is continuous under topology \( S \).
Continuous Functions

**Definition 117 (Continuity at a point)**

Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is **continuous at** $x \in X$ if for every open neighborhood $V$ of $f(x)$ there is an open neighborhood $U$ of $x$ such that $f(U) \subset V$.

**Proposition 118**

Let $X$ and $Y$ be spaces and $f : X \to Y$ be a function. TFAE

1. $f : X \to Y$ is continuous.
2. For all subsets $A \subset X$ we have $f(\overline{A}) \subset \overline{f(A)}$.
3. For all closed sets $B \subset Y$ the set $f^{-1}(B)$ is closed in $X$.
4. $f : X \to Y$ is continuous at $x$ for all $x \in X$.

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**Proof of Prop. 118.**

▶ Claim I: (1) $\implies$ (2)

▶ Suppose $f : X \to Y$ is continuous and let $A \subset X$.
▶ Suppose $y \in f(\overline{A})$.
▶ Then there is $x \in A$ with $f(x) = y$.
▶ Let $V$ be an open neighborhood of $f(x)$.
▶ Then $f^{-1}(V)$ is an open neighborhood of $x$ so it must intersect $A$.
▶ Let $x' \in A \cap f^{-1}(V)$
▶ $f(x') \in f(A)$ and $f(x') \in V$
▶ $f(x') \in f(A) \cap V$
▶ Thus every nbhd $V$ of $f(x)$ intersects $f(A)$.
▶ Hence $y = f(x) \in \overline{f(A)}$
Claim II: (2) \(\Rightarrow\) (3)

Suppose \( f : X \to Y \) is a function and for every subset \( A \subset X \) we have \( f(A) \subset f(\overline{A}) \).

Let \( B \subset Y \) be a closed set.

Let \( A = f^{-1}(B) \).

\( f(A) = f(f^{-1}(B)) \subset B \).

If \( x \in \overline{A} \) then

\[
f(x) \in f(A) \subset f(A) \subset \overline{B} = B
\]

Hence \( x \in f^{-1}(B) = A \).

Thus \( A \subset A \).

Hence \( A = \overline{A} \) must be a closed set.

Thus \( f^{-1}(B) = A \) is closed.

Claim III: (3) \(\Rightarrow\) (1)

Suppose \( f : X \to Y \) is a function and for every closed set \( B \subset Y \) the set \( f^{-1}(B) \) is closed.

Let \( V \subset Y \) be open.

Let \( B = Y - V \) so \( B \) is closed.

\[
f^{-1}(B) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)
\]

By assumption \( f^{-1}(B) \) is closed so \( X - f^{-1}(V) \) is closed.

Hence \( f^{-1}(V) \) is open.
Proof of Prop. 118 (continued).

- Claim IV: $(1) \Rightarrow (4)$
  - Suppose $f : X \to Y$ is continuous and $x \in X$.
  - Then $U = f^{-1}(V)$ is an open neighborhood of $x$ with $f(U) \subseteq V$.
  - Thus $f$ is continuous at $x$ for all $x \in X$.

- Claim V: $(4) \Rightarrow (1)$
  - Suppose $f : X \to Y$ is continuous at $x$ for all $x \in X$.
  - Let $V \subseteq Y$ be open.
  - If $f^{-1}(V)$ is empty we are done so let $x \in f^{-1}(V)$.
  - Then $f(x) \in V$.
  - $f$ is continuous at $X$ so there is a nbhd $U_x$ of $x$ s.t. $f(U_x) \subseteq V$.
  - Hence $U_x \subseteq f^{-1}(V)$
  - Thus $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$
  - Hence $f^{-1}(V)$ is open.
  - It follows that $f$ is continuous.

Definition 119 (Homeomorphism)

Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is a homeomorphism if

- $f : X \to Y$ is a bijection
- $f : X \to Y$ is a continuous.
- $f^{-1} : Y \to X$ is continuous.