Math 5801
General Topology and Knot Theory

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Course Info

Reading for Wednesday, September 26

Chapter 2.20, pgs. 119-126

No homework this week

Midterm 1 Friday, September 28

- Munkres Chapters 1.1-2.19
- ZFC proofs (I'll supply you with all of the axioms)

Product Topology

Much less important is:

Definition 137 (The box topology)

Let $\{A_\alpha\}_{\alpha\in J}$ be a family of topological spaces indexed by the set J and let

$$\pi_\alpha : \prod_{\alpha\in J} A_\alpha \to A_\alpha$$

be the αth projection function. The **box topology** on $\prod_{\alpha\in J} A_\alpha$ has basis

$$\mathcal{B}_{\text{box}} = \left\{ \prod_{\alpha\in J} V_\alpha \,\middle|\, V_\alpha \text{ open in } A_\alpha \right\}$$

- Notice that the box topology is finer than the product topology.
- For finite products of top. spaces they agree.

Product Topology

Example 138 (Box topology fails Prop. 136)

- Recall Prop. 136 says $f : X \to \prod_{\alpha\in J} A_\alpha$ cont. iff $f(a) = (f_\alpha(a))_{\alpha\in J}$ and each f_a cont.
- In general Prop. 136 fails for box topology.
- Let $f : \mathbf{R} \to \mathbf{R}^\omega_{\text{box}}$ be
$$f(t) = (t, t, t, \cdots)$$
- Let $B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots$
- B is open in box topology but
$$\begin{aligned} f^{-1}(B) &= \{t \in \mathbf{R} | f(t) \in B\} \\ &= \{0\} \end{aligned}$$
- So $f^{-1}(B)$ is not open in $\mathbf{R}$.
- Hence f is not cont. even though $f_n(t) = \pi_n(t, t, \cdots) = t$ is cont. for each $n \in \mathbf{Z}_+$.

Product Topology

Proposition 139 (Products of Hausdorff spaces are Hausdorff)

If A_α is Hausdorff for all $\alpha \in J$ then $\prod_{\alpha \in J} A_\alpha$ is Hausdorff with both box and product topologies.

Proof.

Future HW. □

Product Topology

Proposition 140 (Closure of product is product of closures)

If $A_\alpha \subset X_\alpha$ for all $\alpha \in J$ then in the box and product topologies on $\prod_{\alpha \in J} X_\alpha$

$$\overline{\prod_{\alpha \in J} A_\alpha} = \prod_{\alpha \in J} \overline{A}_\alpha$$

Proof.

- Claim I: $\prod_{\alpha \in J} \overline{A}_\alpha \subset \overline{\prod_{\alpha \in J} A_\alpha}$
  - Let $x = (x_\alpha) \in \prod_{\alpha \in J} \overline{A}_\alpha$
  - Let $U = \prod_{\alpha \in J} U_\alpha$ be basis elt. nbdh of x (in either top.)
  - Then $x_\alpha \in \overline{A}_\alpha$ so there is $y_\alpha \in A_\alpha \cap U_\alpha$.
  - Thus we have $(y_\alpha) \in U \cap \prod_{\alpha \in J} A_\alpha$
  - so $x \in \overline{\prod_{\alpha \in J} A_\alpha}$

□

Product Topology

Proof of Prop. 140 *(continued)*.

- Claim II: $\overline{\prod_{\alpha \in J} A_\alpha} \subset \prod_{\alpha \in J} \overline{A}_\alpha$
  - Let $x = (x_\alpha) \in \overline{\prod_{\alpha \in J} A_\alpha}$.
  - Fix $\beta \in J$ and let U_β be an open nbhd of x_β in A_β.
  - Let $U = \pi_\beta^{-1}(U_\beta)$ which is open in the box & product tops.
  - Then we have some $(y_\alpha) \in U \cap \prod_{\alpha \in J} A_\alpha$
  - In particular $y_\beta \in U_\beta \cap A_\beta$.
  - Hence $x_\beta \in \overline{A}_\beta$.
  - It follows that $x \in \prod_{\alpha \in J} \overline{A}_\alpha$

□

Metric Topology

- We've seen that for $\mathbf{R}$ standard topology comes from taking basis

$$\begin{aligned}\mathcal{B} &= \{(a,b) | a,b \in \mathbf{R}\} \\ &= \{(c-\varepsilon, c+\varepsilon) | c \in \mathbf{R}, \varepsilon > 0\}\end{aligned}$$

- In other words, in $\mathbf{R}$ a nbdh of $c \in \mathbf{R}$ is everthing within ε of c.
- What info on a set X do we need to talk about distance?

Definition 141 (Metric)

Let X be a set. A **metric** on X is a function $d : X \times X \to \mathbf{R}$ such that for all $x, y, z \in X$ we have:

1. $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$.
2. (Symmetry) $d(x,y) = d(y,x)$
3. (Triangle Inequality) $d(x,z) \leq d(x,y) + d(y,z)$.

Metric Topology

Examples 142 (Metrics)

1. Let $d_{\mathbf{R}} : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be the function $d_{\mathbf{R}}(a, b) = |b - a|$.
2. Let $d_{\mathbf{R}^2} : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ be the function

$$d_{\mathbf{R}^2}((x_1, x_2), (y_1, y_2)) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$

   This is a metric on $\mathbf{R}^2$
3. For any set X the **Kronecker delta function** $\delta_X : X \times X \to \mathbf{R}$ with

$$\delta_X(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

4. Let $p \in \mathbf{Z}_+$ be prime and define the p-**adic norm** to be $|\frac{ap^n}{b}|_p = p^{-n}$. Then we have the p-adic metric $d_p : \mathbf{Q} \times \mathbf{Q} \to \mathbf{R}$ given by

$$d_p(x, y) = |y - x|_p$$

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Metric Topology

Definition 143 (Metric Topology)

Let $d : X \times X \to \mathbf{R}$ be a metric on a set X. Let $\varepsilon \in \mathbf{R}$ and $x \in X$ the ε-**ball centered at** x is

$$B_\varepsilon(x) = \{b \in X | d(x, b) < \varepsilon\}.$$

The **metric topology** on X has basis

$$\mathcal{B} = \{B_\varepsilon(x) | \varepsilon > 0 \text{ and } x \in X\}.$$

We say the the metric topology in X is **induced by the metric** d.

Proposition 144 (Metric topology is a topology)

If d is a metric on X then the set of ε-balls is a basis for a topology on X.

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Product Topology

Proof of Prop. 144.

Let d be a metric on X and $\mathcal{B} = \{B_\varepsilon(x) | \varepsilon > 0 \text{ and } x \in X\}$.

- Claim I: $X = \bigcup \mathcal{B}$.
  - For all $x \in X$ we have $x \in B_1(x)$ so $X \subset \bigcup_{x \in X} B_1(x) \subset \bigcup \mathcal{B}$.
- Claim II: If $B_1, B_2 \in \mathcal{B}$ and $z \in B_1 \cap B_2$ then there is B_3 with $z \in B_3 \subset B_1 \cap B_2$
  - Let $B_\varepsilon(x)$ and $B_\eta(y)$ be two basis elements
  - Suppose $z \in B_\varepsilon(x) \cap B_\eta(y)$
  - Let $\nu = \min\{\varepsilon - d(x,z), \eta - d(y,z)\}$
  - If $w \in B_\nu(z)$ then

$$\begin{aligned} d(x,w) &\leq d(x,z) + d(z,w) \\ &\leq d(x,z) + \varepsilon - d(x,z) \\ &\leq \varepsilon \end{aligned}$$

  - Hence $B_\nu(z) \subset B_\varepsilon(x)$.
  - Similarly $B_\nu(z) \subset B_\eta(y)$.

□

Metric Topology

Definition 145 (Diameter of a bounded set)

Let X be a metric space with metric d and $A \subset X$. The subset A is **bounded** if there is $M \in \mathbf{R}$ such that for each $a, b \in A$ we have

$$d(a,b) \leq M.$$

The **diameter** of a bounded set A is

$$\operatorname{diam} A = \sup\{d(a,b) | a, b \in A\}$$

Definition 146 (Standard bounded metric)

Let X be a metric space with metric d. The **standard bounded metric** corresponding to d is the metric $\overline{d} : X \times X \to \mathbf{R}$ given by

$$\overline{d}(x,y) \leq \min\{d(x,y), 1\}$$

Metric Topology

Problem 147

Give sufficient properties on a function $s : \mathbf{R} \to \mathbf{R}$ such that for all metric spaces X with metric d we have that $s \circ d$ is also a metric on X.

Metric Topology

Definition 148 (Metrizable Space)

A topological space X is **metrizable** if there is a metric $d : X \times X \to \mathbf{R}$ on X which induces the topology on X.

Examples 149 (Metrizable topologies)

1. The standard topology on $\mathbf{R}$ is induced by the metric $d(x, y) = |y - x|$.
2. The Kronecker delta function $\delta_X : X \times X \to \mathbf{R}$ induces the discrete topology on X. Hence the discrete topology on X is always metrizable.

Metric Topology

Definition 150 (Metrics on $\mathbf{R}^n$)

1. The **euclidean metric** $d : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ on $\mathbf{R}^n$ for $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ is given by

$$d(x, y) = \sqrt{\sum_{k=1}^{n} (y_k - x_k)^2}.$$

2. The **square metric** $d : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ on $\mathbf{R}^n$ for $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ is given by

$$\rho(x, y) = \max_{1 \le k \le n} |y_k - x_k|.$$

Metric Topology

Definition 151 (Metrics on $\mathbf{R}^J$)

1. Let J be a set. The **uniform metric** $\overline{\rho} : \mathbf{R}^J \times \mathbf{R}^J \to \mathbf{R}$ on $\mathbf{R}^J$ for $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ is given by

$$\overline{\rho}(x, y) = \sup_{\alpha \in J} \overline{d}(y_\alpha - x_\alpha).$$

The induced topology is called the **uniform topology on $\mathbf{R}^J$**.

2. For $p \ge 1$ the **ℓ^p-metric** $d : \mathbf{R}^\omega \times \mathbf{R}^\omega \to \mathbf{R}$ on $\mathbf{R}^\omega$ for $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ is given by

$$d(x, y) = \|x - y\|_p = \left(\sum_{k=1}^{\infty} |y_k - x_k|^p \right)^{\frac{1}{p}}$$

The induced topology is called the **ℓ^p-topology on $\mathbf{R}^\omega$**.