# Math 5801
## General Topology and Knot Theory

Nathan Broaddus
Ohio State University
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## Course Info

### Reading for Monday, October 3

Chapter 2.21, pgs. 129-133

### No homework this week

### Midterm 1 Friday, September 28

- Munkres Chapters 1.1-2.19
- ZFC proofs (I'll supply you with all of the axioms)
Product Topology

Much less important is:

**Definition 137 (The box topology)**

Let \( \{A_\alpha\}_{\alpha \in J} \) be a family of topological spaces indexed by the set \( J \) and let

\[
\pi_\alpha : \prod_{\alpha \in J} A_\alpha \to A_\alpha
\]

be the \( \alpha \)th projection function. The **box topology** on \( \prod_{\alpha \in J} A_\alpha \) has basis

\[
\mathcal{B}_{\text{box}} = \left\{ \prod_{\alpha \in J} V_\alpha \mid V_\alpha \text{ open in } A_\alpha \right\}
\]

- Notice that the box topology is finer than the product topology.
- For finite products of top. spaces they agree.

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**Example 138 (Box topology fails Prop. 136)**

- Recall Prop. 136 says \( f : X \to \prod_{\alpha \in J} A_\alpha \) cont. iff \( f(a) = (f_\alpha(a))_{\alpha \in J} \) and each \( f_\alpha \) cont.
- In general Prop. 136 fails for box topology.
- Let \( f : \mathbb{R} \to \mathbb{R}^\omega \) be

\[
f(t) = (t, t, t, \cdots)
\]

- Let \( B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots \)
- \( B \) is open in box topology but

\[
f^{-1}(B) = \{ t \in \mathbb{R} | f(t) \in B \} = \{0\}
\]

- So \( f^{-1}(B) \) is not open in \( \mathbb{R} \).
- Hence \( f \) is not cont. even though \( f_n(t) = \pi_n(t, t, \cdots) = t \) is cont. for each \( n \in \mathbb{Z}_+ \).
Proposition 139 (Products of Hausdorff spaces are Hausdorff)

If $A_{\alpha}$ is Hausdorff for all $\alpha \in J$ then \( \prod_{\alpha \in J} A_{\alpha} \) is Hausdorff with both box and product topologies.

Proof.
Future HW.

Proposition 140 (Closure of product is product of closures)

If $A_{\alpha} \subset X_{\alpha}$ for all $\alpha \in J$ then in the box and product topologies on \( \prod_{\alpha \in J} X_{\alpha} \)

\[
\bigcap_{\alpha \in J} A_{\alpha} \subset \bigcap_{\alpha \in J} \overline{A}_{\alpha}
\]

Proof.

Claim I: \( \bigcap_{\alpha \in J} A_{\alpha} \subset \bigcap_{\alpha \in J} \overline{A}_{\alpha} \)

- Let \( x = (x_{\alpha}) \in \prod_{\alpha \in J} \overline{A}_{\alpha} \)
- Let \( U = \prod_{\alpha \in J} U_{\alpha} \) be basis elt. nbdh of \( x \) (in either top.)
- Then \( x_{\alpha} \in \overline{A}_{\alpha} \) so there is \( y_{\alpha} \in A_{\alpha} \cap U_{\alpha} \).
- Thus we have \( (y_{\alpha}) \in U \cap \prod_{\alpha \in J} A_{\alpha} \)
- so \( x \in \prod_{\alpha \in J} A_{\alpha} \)
Proof of Prop. 140 (continued).

- **Claim II:** $\prod_{\alpha \in J} A_\alpha \subset \prod_{\alpha \in J} \overline{A_\alpha}$
  - Let $x = (x_\alpha) \in \prod_{\alpha \in J} A_\alpha$.
  - Fix $\beta \in J$ and let $U_\beta$ be an open nbhd of $x_\beta$ in $A_\beta$.
  - Let $U = \pi_\beta^{-1}(U_\beta)$ which is open in the box & product tops.
  - Then we have some $(y_\alpha) \in U \cap \prod_{\alpha \in J} A_\alpha$.
  - In particular $y_\beta \in U_\beta \cap A_\beta$.
  - Hence $x_\beta \in \overline{A_\beta}$.
  - It follows that $x \in \prod_{\alpha \in J} \overline{A_\alpha}$.

Metric Topology

- We’ve seen that for $\mathbb{R}$ standard topology comes from taking basis

  $\mathcal{B} = \{(a, b) | a, b \in \mathbb{R}\}$

  $\mathcal{B} = \{(c - \varepsilon, c + \varepsilon) | c \in \mathbb{R}, \varepsilon > 0\}$

- In other words, in $\mathbb{R}$ a nbhd of $c \in \mathbb{R}$ is everything within $\varepsilon$ of $c$.
- What info on a set $X$ do we need to talk about distance?

**Definition 141 (Metric)**

Let $X$ be a set. A **metric** on $X$ is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$ we have:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$.
2. (Symmetry) $d(x, y) = d(y, x)$
3. (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$.
**Metric Topology**

### Examples 142 (Metrics)

1. Let $d_R : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function $d_R(a, b) = |b - a|$.
2. Let $d_{R^2} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the function
   
   $$d_{R^2}((x_1, x_2), (y_1, y_2)) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$  
   This is a metric on $\mathbb{R}^2$.
3. For any set $X$ the **Kronecker delta function** $\delta_X : X \times X \to \mathbb{R}$ with
   
   $$\delta_X(x, y) = \begin{cases} 
   0, & x = y \\
   1, & x \neq y 
   \end{cases}$$
4. Let $p \in \mathbb{Z}_+$ be prime and define the **$p$-adic norm** to be $|\frac{a^p}{b}|_p = p^{-n}$. Then we have the **$p$-adic metric** $d_p : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ given by
   
   $$d_p(x, y) = |y - x|_p$$

### Definition 143 (Metric Topology)

Let $d : X \times X \to \mathbb{R}$ be a metric on a set $X$. Let $\varepsilon \in \mathbb{R}$ and $x \in X$ the **$\varepsilon$-ball centered at $x$** is

$$B_\varepsilon(x) = \{ b \in X | d(x, b) < \varepsilon \}.$$  

The **metric topology** on $X$ has basis

$$\mathcal{B} = \{ B_\varepsilon(x) | \varepsilon > 0 \text{ and } x \in X \}.$$  

We say the the metric topology on $X$ is **induced by the metric $d$**.

### Proposition 144 (Metric topology is a topology)

*If $d$ is a metric on $X$ then the set of $\varepsilon$-balls is a basis for a topology on $X$.*
Product Topology

Proof of Prop. 144.

Let \( d \) be a metric on \( X \) and \( \mathcal{B} = \{ B_\varepsilon(x) \mid \varepsilon > 0 \text{ and } x \in X \} \).

- **Claim I:** \( X = \bigcup \mathcal{B} \).
  - For all \( x \in X \) we have \( x \in B_\varepsilon(x) \) so \( X \subset \bigcup_{x \in X} B_\varepsilon(x) \subset \bigcup \mathcal{B} \).

- **Claim II:** If \( B_\varepsilon \), \( B_\eta \in \mathcal{B} \) and \( z \in B_\varepsilon \cap B_\eta \) then there is \( B_\nu \) with \( z \in B_\nu \subset B_\varepsilon \cap B_\eta \).
  - Let \( B_\varepsilon(x) \) and \( B_\eta(y) \) be two basis elements.
  - Suppose \( z \in B_\varepsilon(x) \cap B_\eta(y) \).
  - Let \( \nu = \min\{ \varepsilon - d(x, z), \eta - d(y, z) \} \).
  - If \( w \in B_\nu(z) \) then
    \[
    d(x, w) \leq d(x, z) + d(z, w) \\
    \leq d(x, z) + \varepsilon - d(x, z) \\
    \leq \varepsilon
    \]
  - Hence \( B_\nu(z) \subset B_\varepsilon(x) \).
  - Similarly \( B_\nu(z) \subset B_\eta(y) \).

Metric Topology

Lemma 145

*If \( X \) is a metric space and \( U \) is an open nbhd of \( x \in X \) then there is some \( \varepsilon > 0 \) such that \( x \in B_\varepsilon(x) \subset U \).*

**Proof.**

- Let \( U \) be an open nbhd of \( x \in X \) with metric \( d \).
- Then there is some \( y \in X \) and \( \eta > 0 \) such that \( x \in B_\eta(y) \subset U \).
- Let \( \varepsilon = \eta - d(x, y) \).
- Then \( \varepsilon > 0 \) since \( d(x, y) \geq 0 \).
- Also, if \( z \in B_\varepsilon(x) \) then
  \[
  d(y, z) \leq d(y, x) + d(x, z) \\
  \leq d(y, x) + \eta - d(x, y) \\
  = \eta
  \]
- So \( B_\varepsilon(x) \subset B_\eta(y) \subset U \).
Metric Topology

Definition 146 (Diameter of a bounded set)
Let $X$ be a metric space with metric $d$ and $A \subset X$. The subset $A$ is **bounded** if there is $M \in \mathbb{R}$ such that for each $a, b \in A$ we have
\[ d(a, b) \leq M. \]
The **diameter** of a bounded set $A$ is
\[ \text{diam } A = \sup \{ d(a, b) | a, b \in A \} \]

Definition 147 (Standard bounded metric)
Let $X$ be a metric space with metric $d$. The **standard bounded metric** corresponding to $d$ is the metric $\overline{d} : X \times X \to \mathbb{R}$ given by
\[ \overline{d}(x, y) = \min \{ d(x, y), 1 \} \]

Proposition 148
Let $X$ be a metric space with metric $d$. Then standard bounded metric $\overline{d} : X \times X \to \mathbb{R}$ given by
\[ \overline{d}(x, y) = \min \{ d(x, y), 1 \} \]
is a metric on $X$.

**Proof.**
Let $d$ be a metric on $X$ and $\overline{d} : X \times X \to \mathbb{R}$ be the standard bounded metric corresponding to $d$.

- **Claim I:** If $x, y \in X$ then $\overline{d}(x, y) \geq 0$.
  - Let $x, y \in X$.
  - Then
    \[ \overline{d}(x, y) = \min \{ d(x, y), 1 \} \geq \min \{ 0, 1 \} = 0 \]
Metric Topology

Proof of Prop. 148 (continued).

▶ Claim II: If \( x, y \in X \) and \( \overline{d}(x, y) = 0 \) then \( x = y \).
  ▶ Let \( x, y \in X \) and suppose \( \overline{d}(x, y) = 0 \).
  ▶ Then \( \overline{d}(x, y) = 0 \) so \( x = y \).

▶ Claim III: If \( x, y \in X \) and \( \overline{d}(x, y) = \overline{d}(y, x) \)
  ▶ Let \( x, y \in X \).
  ▶ Then

\[
\overline{d}(x, y) = \min\{d(x, y), 1\} \\
= \min\{d(y, x), 1\} \\
= \overline{d}(y, x)
\]

▶ Claim IV: If \( x, y, z \in X \) then \( \overline{d}(x, z) \leq \overline{d}(x, y) + \overline{d}(y, z) \).
  ▶ Let \( x, y, z \in X \)
  ▶ Case A: \( d(x, y) \leq 1 \) and \( d(y, z) \leq 1 \).

\[
\overline{d}(x, z) \leq d(x, z) \\
\leq d(x, y) + d(y, z) \\
= \overline{d}(x, y) + \overline{d}(y, z)
\]

  ▶ Case B: \( d(x, y) > 1 \) or \( d(y, z) > 1 \).

\[
\overline{d}(x, z) \leq 1 \\
\leq \overline{d}(x, y) + \overline{d}(y, z)
\]
Problem 149

Give sufficient properties on a function \( s : \mathbb{R} \to \mathbb{R} \) such that for all metric spaces \( X \) with metric \( d \) we have that \( s \circ d \) is also a metric on \( X \).

Solution

- Suppose \( s : \mathbb{R} \to \mathbb{R} \) satisfies:
  1. \( s(0) = 0 \)
  2. For \( a > 0 \) we have \( s(a) > 0 \).
  3. \( s \) is nondecreasing
  4. \( s \) is convex (For all \( a, b \geq 0 \) we have \( s(a + b) \leq s(a) + s(b) \)).

Then \( s \circ d \) will be a metric.

- Let \( x, y, z \in X \). Then

\[
\begin{align*}
    s \circ d(x, z) &\leq s(d(x, y) + d(y, z)) \quad \text{by (2)} \\
    &\leq s \circ d(x, y) + s \circ d(y, z) \quad \text{by (3)}
\end{align*}
\]

- Reinterpret Prop. 148 as a convexity problem.

Definition 150 (Metrizable Space)

A topological space \( X \) is **metrizable** if there is a metric \( d : X \times X \to \mathbb{R} \) on \( X \) which induces the topology on \( X \).

Examples 151 (Metrizable topologies)

1. The standard topology on \( \mathbb{R} \) is induced by the metric \( d(x, y) = |y - x| \).
2. The Kronecker delta function \( \delta_X : X \times X \to \mathbb{R} \) induces the discrete topology on \( X \). Hence the discrete topology on \( X \) is always metrizable.
Metric Topology

Definition 152 (Metrics on $\mathbb{R}^n$)

1. The euclidean metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$ for $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ is given by

$$d(x, y) = \sqrt{n \sum_{k=1}^{n} (y_k - x_k)^2}.$$

2. The square metric $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$ for $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ is given by

$$\rho(x, y) = \max_{1 \leq k \leq n} |y_k - x_k|.$$

The induced topology is called the uniform topology on $\mathbb{R}^n$.

Definition 153 (Metrics on $\mathbb{R}^J$)

1. Let $J$ be a set. The uniform metric $\bar{p} : \mathbb{R}^J \times \mathbb{R}^J \to \mathbb{R}$ on $\mathbb{R}^J$ for $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ is given by

$$\bar{p}(x, y) = \sup_{\alpha \in J} |y_\alpha - x_\alpha|.$$

The induced topology is called the uniform topology on $\mathbb{R}^J$.

2. For $p \geq 1$ the $\ell^p$-metric $d : \mathbb{R}^\omega \times \mathbb{R}^\omega \to \mathbb{R}$ on $\mathbb{R}^\omega$ for $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ is given by

$$d(x, y) = \|x - y\|_{\ell^p} = \left( \sum_{k=1}^{\infty} |y_k - x_k|^p \right)^{1/p}.$$

The induced topology is called the $\ell^p$-topology on $\mathbb{R}^\omega$. 
Metric Topology

**Proposition 154 (Metric spaces are Hausdorff)**

If $X$ is a metric space then its induced topology is Hausdorff.

**Proof.**

- Let $d : X \times X \to \mathbb{R}$ be a metric on $X$.
- Suppose $x, y \in X$ and $x \neq y$.
- Let $\varepsilon = d(x, y) > 0$.
- Claim: $U_x = B_{\frac{\varepsilon}{2}}(x)$ and $U_y = B_{\frac{\varepsilon}{2}}(y)$ are disjoint open nbhds of $x$ and $y$ resp. with $U_x \cap U_y = \emptyset$.
- Suppose $U_x \cap U_y \neq \emptyset$.
- Then there is $z \in U_x \cap U_y$.
- Then $d(x, y) = \varepsilon$ by $d(x, z) < \frac{\varepsilon}{2}$ and $d(z, y) < \frac{\varepsilon}{2}$ so
- $d(x, y) > d(x, z) + d(z, y)$ contradicting triangle ineq.

Example 155 (A non-metrizable topology)

$\mathbb{Z}_f$ is not Hausdorff so by Prop. 157 it in not metrizable.

**Definition 156 ($\delta, \varepsilon$)-continuous**

Let $X$ and $Y$ be metric spaces. A function $f : X \to Y$ is $(\delta, \varepsilon)$-continuous if for every $a \in X$ and every $\varepsilon > 0$ there is $\delta > 0$ such that for every $b \in X$ $d_X(a, b) < \delta$ implies $d_Y(f(a), f(b)) < \varepsilon$.

**Proposition 157 ($(\delta, \varepsilon)$-continuous is equivalent to continuous)**

Let $X$ and $Y$ be metric spaces. A function $f : X \to Y$ is continuous in the induced metric topologies if and only if it is $(\delta, \varepsilon)$-continuous.

**Proof.**

Proof of Prop. 115 easy to rework here.
Metric Topology

- How much do sequences in metric spaces have in common with sequences in $\mathbb{R}$?
- For example, we saw that $(\frac{1}{n})_{n \in \mathbb{Z}^+}$ converges to $\pi$ in $\mathbb{R}$.
- Metric spaces are Hausdorff so at least limits of sequences are unique.
- For a subset $A \subset \mathbb{R}$, $x \in \overline{A}$ iff there is a sequence in $A$ which converges to $x$.
- One direction is true for all top. spaces:

**Proposition 158**

Let $X$ be a topological space and $A \subset X$. Then if there is a sequence $(a_n)_{n \in \mathbb{Z}^+}$ in $A$ which converges to $x$ then $x \in \overline{A}$

**Proof.**

If $a_n \to x$ then every nbdh of $x$ contains an element of $A$. So $x \in \overline{A}$. □

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**Proposition 159**

Let $X$ be a metric space and $A \subset X$. Then there is a sequence $(a_n)_{n \in \mathbb{Z}^+}$ in $A$ which converges to $x$ if and only if $x \in \overline{A}$

**Proof.**

- $\Rightarrow$ follows from Prop. 158 above.
- Suppose $x \in \overline{A}$.
- Then every nbdh $U_x$ of $x$ intersects $A$.
- For each $n \in \mathbb{Z}^+$ choose $a_n \in A \cap B_{\frac{1}{n}}(x)$
- Claim: $a_n \to x$.
- Let $U_x$ be an open nbdh of $x$.
- Then there is some $\varepsilon > 0$ s.t. $B_{\varepsilon}(x) \subset U_x$
- Choose $N \in \mathbb{Z}^+$ large so that $\frac{1}{N} < \varepsilon$.
- Then for all $n > N$ we have $a_n \in B_{\frac{1}{n}}(x) \subset B_{\varepsilon}(x) \subset U_x$. □