Course Info

Reading for Monday, October 8
Chapter 3.23, pgs. 147-152

HW 6 for Monday, October 8
- Chapter 2.19: 3, 6, 7, 8
- Chapter 2.20: 2, 3a-b, 11
Quotient Topology

**Definition 179 (The unit interval)**

The **unit interval** $I$ is the topological (or metric) space $I = [0, 1]$ with the subspace topology (resp. subset metric) inherited from $\mathbb{R}$.

**Example 180 (The 2-torus $T^2$)**

- Let $\sim$ on $I^2$ be the relation with $(0, a) \sim (1, a)$ for and $(a, 0) \sim (a, 1)$ for all $a \in I$.
- The **2-torus** is the quotient space $T^2 = I^2 / \sim$.
- Let $q : I^2 \to T^2$ be the quotient map $q(x, y) = [x, y]$ where $[x, y]$ is the equiv. class of $(x, y) \in I^2$ under $\sim$.
- Let $0 < a, b < 1$. What does a nbhd of $[a, b] \in T^2$ look like?
  - Let $\varepsilon = \min\{|a - 0|, |a - 1|, |b - 0|, |b - 1|\}$
  - Then for each $p \in B_\varepsilon(a, b) \subset I^2$ we have $[p] = \{p\}$.
  - Thus $B_\varepsilon(a, b) \subset I^2$ is a saturated set.
  - And $q|_{B_\varepsilon(a, b)} : B_\varepsilon(a, b) \to B_\varepsilon(a, b)$ is an injection.
  - Hence by def. of quot. top. $q|_{B_\varepsilon(a, b)}$ is a homeomorphism.

**Example 181 (The $n$-torus $T^n$)**

- Let $\sim$ on $\mathbb{R}^n$ be the relation with $(x, y) \sim (x + m, y + n)$ for all $x, y \in \mathbb{R}$ and $n, m \in \mathbb{R}$
- The **$n$-torus** is the quotient space $T^n = \mathbb{R}^n / \sim$.
- What do open sets of $T^n$ look like?
- Claim: If $U \subset \mathbb{R}^n$ is open then $q(U)$ is open in $T^n$
  - If $U \subset \mathbb{R}^n$ is open then the smallest saturated set containing $U$ is $q^{-1}(q(U))$
  - $q^{-1}(q(U)) = \bigcup_{(x, y) \in U} [x, y]$
  - $= \bigcup_{(x, y) \in U} \{(x + m, y + n) | n, m \in \mathbb{Z}\}$
  - $= \bigcup_{n, m \in \mathbb{Z}} \{(x + m, y + n) | (x, y) \in U\}$
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**Quotient Topology**

**Example 182**

- Let $\sim$ on $\mathbb{R}$ be the relation with $x \sim y$ if $x - y \in \mathbb{Q}$.
- Let $Y = \mathbb{R}/\sim$ with quotient map $q : \mathbb{R} \to Y$.
- Notice that for each $y \in Y$ we have $q^{-1}(y)$ id countable (since $\mathbb{Q}$ is countable).
- Thus $Y$ is uncountable (otherwise $\mathbb{R}$ would be a countable union of countable sets).
- If nonempty $U \subset \mathbb{R}$ is open then the smallest saturated set containing $U$ is
  
  $$q^{-1}(q(U)) = \bigcup_{x \in U} [x] = \mathbb{R}$$

- Thus $Y$ is an uncountable set with the trivial topology.

**Quotient Topology**

**Definition 183 (Disjoint union and Coproduct Topology)**

If $A$ and $B$ are sets then the **disjoint union** of $A$ and $B$ is the set

$$A \amalg B = A \times \{0\} \cup B \times \{1\}$$

We write $A \subset A \amalg B$ and $B \subset A \amalg B$ even though not strictly true.

The **disjoint union** of the collection of indexed sets $\{A_\alpha\}_{\alpha \in A}$ is

$$\coprod_{\alpha \in A} A_\alpha = \bigcup_{\alpha \in A} A_\alpha \times \{\alpha\}.$$

We identify $A_\alpha$ with $A_\alpha \times \{\alpha\} \subset \coprod_{\alpha \in A} A_\alpha$.

If each $A_\alpha$ is a topological space then $\coprod_{\alpha \in A} A_\alpha$ is given the **coproduct topology** where $B \subset \coprod_{\alpha \in A} A_\alpha$ is open if $B \cap A_\alpha$ is open for all $\alpha \in A$. 
Quotient Topology

- Let $X$ be a topological space and $Y$ a set of topological spaces.
- Let $f : X \to \prod Y$ be a function and for each $Y \in Y$ let $f_Y : X \to Y$ be $\pi_Y \circ f$.
- Recall that Prop. 136 says that $f$ is continuous iff each $f_Y$ is continuous.
- Compare to the following:

**Proposition 184 (Universal property of coproduct)**

Let $X$ be a set of topological spaces and $Y$ be a topological space. A function $f : \coprod X \to Y$ is continuous if and only if $f|_X : X \to Y$ is continuous for each $X \in X$.

**Proof.**

Immediate from def. of open sets of $\coprod X$.

Example 185

- Let $\sim$ on $\mathbb{R}$ be the relation with $x \sim 2^n x$ if $n \in \mathbb{Z}$.
- Let $W = \mathbb{R}/\sim$ with quotient map $q : \mathbb{R} \to W$.
- If $x \neq 0$ then $[x]$ has a nbhd homeomorphic to $S^1$.
- Only saturated open set containing $[0]$ is $\mathbb{R}$.
- Hence $W \cong S^1 \sqcup S^1 \sqcup \{[0]\}$ where $[0]$ is in the closure of every nonempty subset of $W$. 
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Quotient Topology

**Definition 186 (Graph)**

1. Let $X^0$ be a set with the discrete topology called the **vertices**.
2. Let $\mathcal{A}$ be a set and for each
3. For each $\alpha \in \mathcal{A}$ let $I_\alpha = I = [0, 1]$.
4. For each $\alpha \in \mathcal{A}$ let $\varphi_\alpha : \partial I_\alpha \to X^0$ where $\partial I_\alpha = \{0, 1\} \subset I_\alpha$.
5. Let
   $$Y = X^0 \amalg \bigsqcup_{\alpha \in \mathcal{A}} I_\alpha$$
6. Let $\sim$ on $Y$ be the equivalence relation $x \sim y$ if
   - $x \in I_\alpha$ and $y \in I_\beta$ and $\phi_\alpha(x) = \phi_\beta(y)$.
   - or $x \in I_\alpha$ and $y \in X^0$ and $\phi_\alpha(x) = y$.
7. A **graph** is the quotient $X^1 = Y / \sim$.
8. Let $q : Y \to X^1$ be the quotient map.
9. Let $i_\alpha : I_\alpha \to Y$ be the inclusion.
10. An **edge** of the graph $X^1$ is a set of the form $q \circ i_\alpha(I_\alpha)$.

**Proposition 187 (Universal Property of Quotient Space)**

1. Let $X$ be a topological space and $\sim$ be an equivalence relation on $X$.
2. Let $Z$ be a topological space.
3. Let $g : X \to Z$ be a continuous function.
   - There is a continuous function $f : (X/ \sim) \to Z$ such that $g = f \circ q$ if for each $[x] \in X/ \sim$ we have that $g_{|[x]} : \{[x]\}$ is constant.
Proof of Prop. 187.

- Let $X$ be a topological space and $\sim$ be an equivalence relation on $X$ with quotient map $q : X \to X/\sim$
- Let $Z$ be a topological space.
- Let $g : X \to Z$ be a continuous function.

$\Rightarrow$
- Suppose there is continuous $f : (X/\sim) \to Z$ such that $g = f \circ q$.
- If $x_0, x_1 \in X$ and $x_0 \sim x_1$ then
  
  $$g(x_0) = f \circ q(x_0) = f([x_0]) = f([x_1]) = f \circ q(x_1) = g(x_1)$$

Proof of Prop. 187 (continued).

$\Leftarrow$
- Suppose that for all $x_0, x_1 \in X$ if $x_0 \sim x_1$ then $g(x_0) = g(x_1)$
- Define $f : X/\sim \to Z$ to be the function $f([x]) = g(x)$.
- By assumption $f$ is well-defined.
- If $V \subset Z$ is open then $g^{-1}(V)$ is open by continuity of $g$.
- So we have open
  
  $$g^{-1}(V) = (f \circ q)^{-1}(V) = q^{-1}(f^{-1}(V))$$

- By def. of quotient top. $f^{-1}(V)$ is open in $X/\sim$. 

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