Math 5801
General Topology and Knot Theory

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Course Info

Reading for Monday, October 8

Chapter 3.23, pgs. 147-152

HW 6 for Monday, October 8

- Chapter 2.19: 3, 6, 7, 8
- Chapter 2.20: 2, 3a-b, 11

Quotient Topology

Definition 179 (The unit interval)

The **unit interval** I is the topological (or metric) space $I = [0, 1]$ with the subspace topology (resp. subset metric) inherited from $\mathbf{R}$.

Example 180 (The 2-torus T^2)

- Let $\sim$ on I^2 be the relation with $(0, a) \sim (1, a)$ for and $(a, 0) \sim (a, 1)$ for all $a \in I$.
- The 2-**torus** is the quotient space $T^2 = I^2/\sim$.
- Let $q : I^2 \to T^2$ be the quotient map $q(x, y) = [x, y]$ where $[x, y]$ is the equiv. class of $(x, y) \in I^2$ under $\sim$.
- Let $0 < a, b < 1$. What does a nbhd of $[a, b] \in T^2$ look like?
  - Let $\varepsilon = \min\{|a - 0|, |a - 1|, |b - 0|, |b - 1|\}$
  - Then for each $p \in B_\varepsilon(a, b) \subset I^2$ we have $[p] = \{p\}$.
  - Thus $B_\varepsilon(a, b) \subset I^2$ is a saturated set.
  - And $q|_{B_\varepsilon(a,b)} : B_\varepsilon(a, b) \to B_\varepsilon(a, b)$ is an injection.
  - Hence by def. of quot. top. $q|_{B_\varepsilon(a,b)}$ is a homeomorphism.

Quotient Topology

Example 181 (The n-torus T^n)

- Let $\sim$ on $\mathbf{R}^n$ be the relation with $(x, y) \sim (x + m, y + n)$ for all $x, y \in \mathbf{R}$ and $n, m \in \mathbf{R}$
- The n-**torus** is the quotient space $T^n = \mathbf{R}^n/\sim$.
- What do open sets of T^n look like?
- Claim: If $U \subset \mathbf{R}^n$ is open then $q(U)$ is open in T^n
  - If $U \subset \mathbf{R}^n$ is open then the smallest saturated set containing U is $q^{-1}(q(U))$
  - $$\begin{aligned} q^{-1}(q(U)) &= \bigcup_{(x,y)\in U} [x, y] \\ &= \bigcup_{(x,y)\in U} \{(x + m, y + n) | n, m \in \mathbf{Z}\} \\ &= \bigcup_{n,m\in\mathbf{Z}} \{(x + m, y + n) | (x, y) \in U\} \end{aligned}$$

Quotient Topology

Example 182

- Let $\sim$ on $\mathbf{R}$ be the relation with $x \sim y$ if $x - y \in \mathbf{Q}$.
- Let $Y = \mathbf{R}/\sim$ with quotient map $q : \mathbf{R} \to Y$.
- Notice that for each $y \in Y$ we have $q^{-1}(y)$ id countable (since $\mathbf{Q}$ is countable)
- Thus Y is uncountable (otherwise $\mathbf{R}$ would be a countable union of countable sets).
- If nonempty $U \subset \mathbf{R}$ is open then the smallest saturated set containg U is

$$\begin{aligned} q^{-1}(q(U)) &= \bigcup_{x \in U} [x] \\ &= \mathbf{R} \end{aligned}$$

- Thus Y is an uncountable set with the trivial topology.

Quotient Topology

Definition 183 (Disjoint union and Coproduct Topology)

If A and B are sets then the **disjoint union** of A and B is the set

$$A \amalg B = A \times \{0\} \cup B \times \{1\}$$

We write $A \subset A \amalg B$ and $B \subset A \amalg B$ even thought not strictly true.

The **disjoint union** of the collection of indexed sets $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ is

$$\coprod_{\alpha \in \mathcal{A}} A_\alpha = \bigcup_{\alpha \in \mathcal{A}} A_\alpha \times \{\alpha\}.$$

We identify A_α with $A_\alpha \times \{\alpha\} \subset \coprod_{\alpha \in \mathcal{A}} A_\alpha$.

If each A_α is a topological space then $\coprod_{\alpha \in \mathcal{A}} A_\alpha$ is given the **coproduct topology** where $B \subset \coprod_{\alpha \in \mathcal{A}} A_\alpha$ is open iff $B \cap A_\alpha$ is open for all $\alpha \in \mathcal{A}$.

Quotient Topology

- Let X be a topological space and $\mathcal{Y}$ a set of topological spaces.
- Let $f : X \to \prod \mathcal{Y}$ be a function and for each $Y \in \mathcal{Y}$ let $f_Y : X \to Y$ be $\pi_Y \circ f$.
- Recall that Prop. 136 says that f is continuous iff each f_Y is continuous.
- Compare to the following:

Proposition 184 (Universal property of coproduct)

Let $\mathcal{X}$ be a set of topological spaces and Y be a topological space. A function $f : \coprod \mathcal{X} \to Y$ is continuous if and only if $f|_X : X \to Y$ is continuous for each $X \in \mathcal{X}$.

Proof.

Immediate from def. of open sets of $\coprod \mathcal{X}$ □

Quotient Topology

Example 185

- Let $\sim$ on $\mathbf{R}$ be the relation with $x \sim 2^n x$ if $n \in \mathbf{Z}$.
- Let $W = \mathbf{R}/\sim$ with quotient map $q : \mathbf{R} \to W$.
- If $x \neq 0$ then $[x]$ has a nbhd homeomorphic to S^1.
- Only saturated open set containing $[0]$ is $\mathbf{R}$.
- Hence $W \cong S^1 \amalg S^1 \cup \{[0]\}$ where $[0]$ is in the closure of every nonempty subset of W.

Quotient Topology

Definition 186 (Graph)

1. Let X^0 be a set with the discrete topology called the **vertices**.
2. Let $\mathcal{A}$ be a set and for each
3. For each $\alpha \in \mathcal{A}$ let $I_\alpha = I = [0,1]$.
4. For each $\alpha \in \mathcal{A}$ let $\varphi_\alpha : \partial I_\alpha \to X^0$ where $\partial I_\alpha = \{0,1\} \subset I_\alpha$
5. Let
$$Y = X^0 \amalg \coprod_{\alpha \in \mathcal{A}} I_\alpha$$
6. Let $\sim$ on Y be the equivalence relation $x \sim y$ if
   - $x \in I_\alpha$ and $y \in I_\beta$ and $\phi_\alpha(x) = \phi_\beta(y)$.
   - or $x \in I_\alpha$ and $y \in X^0$ and $\phi_\alpha(x) = y$.
7. A **graph** is the quotient $X^1 = Y/\sim$.
8. Let $q : Y \to X^1$ be the quotient map.
9. Let $i_\alpha : I_\alpha \to Y$ be the inclusion
10. An **edge** of the graph X^1 is a set of the form $q \circ i_\alpha(I_\alpha)$.

Quotient Topology

- Let X be a topological space and $\sim$ be an equivalence relation on X.
- Let Z be a topological space.
- What are the continuous functions $f : (X/\sim) \to Z$?
- We have that the quotient map $q : X \to X/\sim$ is continuous so any continuous function $f : (X/\sim) \to Z$ gives continuous $f \circ q : X \to Z$.
- For which functions $g : X \to Z$ is there a function $f : (X/\sim) \to Z$ such that $g = f \circ q$?

Proposition 187 (Universal Property of Quotient Space)

1. *Let X be a topological space and $\sim$ be an equivalence relation on X with quotient map $q : X \to X/\sim$*
2. *Let Z be a topological space.*
3. *Let $g : X \to Z$ be a continuous function.*

There is a continuous function $f : (X/\sim) \to Z$ such that $g = f \circ q$ iff for each $[x] \in X/\sim$ we have that $g|_{q^{-1}}(\{[x]\})$ is constant.

Quotient Topology

Proof of Prop. 187.

- Let X be a topological space and $\sim$ be an equivalence relation on X with quotient map $q : X \to X/\sim$
- Let Z be a topological space.
- Let $g : X \to Z$ be a continuous function.
- $\Rightarrow$
  - Suppose there is continuous $f : (X/\sim) \to Z$ such that $g = f \circ q$.
  - If $x_0, x_1 \in X$ and $x_0 \sim x_1$ then

$$g(x_0) = f \circ q(x_0) = f([x_0]) = f([x_1]) = f \circ q(x_1) = g(x_1)$$

Quotient Topology

Proof of Prop. 187 *(continued)*.

- $\Leftarrow$
  - Suppose that for all $x_0, x_1 \in X$ if $x_0 \sim x_1$ then $g(x_0) = g(x_1)$
  - Define $f : X/\sim \to Z$ to be the function $f([x]) = g(x)$.
  - By assumption f is well-defined.
  - If $V \subset Z$ is open then $g^{-1}(V)$ is open by continuity of g.
  - So we have open

$$g^{-1}(V) = (f \circ q)^{-1}(V) = q^{-1}(f^{-1}(V))$$

  - By def. of quotient top. $f^{-1}(V)$ is open in $X/\sim$.

$\square$