Math 5801
General Topology and Knot Theory

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Course Info

Reading for Wednesday, October 10
Review Chapter 3.23

HW 7 for Monday, October 15
- Chapter 2.21: 7, 8, 12b
- Chapter 2.22: 2a-b, 4a, 5
- Chapter 3.23: 2, 4, 11
Quotient Topology

Let $X$ be a topological space and $\sim$ be an equivalence relation on $X$.

Let $Z$ be a topological space.

What are the continuous functions $f : (X / \sim) \to Z$?

We have that the quotient map $q : X \to X / \sim$ is continuous so any continuous function $f : (X / \sim) \to Z$ gives continuous $f \circ q : X \to Z$.

For which functions $g : X \to Z$ is there a function $f : (X / \sim) \to Z$ such that $g = f \circ q$?

**Proposition 187 (Universal Property of Quotient Space)**

1. Let $X$ be a topological space and $\sim$ be an equivalence relation on $X$ with quotient map $q : X \to X / \sim$.
2. Let $Z$ be a topological space.
3. Let $g : X \to Z$ be a continuous function.

There is a continuous function $f : (X / \sim) \to Z$ such that $g = f \circ q$ iff for each $[x] \in X / \sim$ we have that $g|_q^{-1}([x])$ is constant.

**Proof of Prop. 187.**

Let $X$ be a topological space and $\sim$ be an equivalence relation on $X$ with quotient map $q : X \to X / \sim$.

Let $Z$ be a topological space.

Let $g : X \to Z$ be a continuous function.

Suppose there is continuous $f : (X / \sim) \to Z$ such that $g = f \circ q$.

If $x_0, x_1 \in X$ and $x_0 \sim x_1$ then

$$g(x_0) = f \circ q(x_0) = f([x_0]) = f([x_1]) = f \circ q(x_1) = g(x_1)$$
Proof of Prop. 187 (continued).

\[ \Leftarrow \]

- Suppose that for all \( x_0, x_1 \in X \) if \( x_0 \sim x_1 \) then \( g(x_0) = g(x_1) \)
- Define \( f : X/\sim \to Z \) to be the function \( f([x]) = g(x) \).
- By assumption \( f \) is well-defined.
- If \( V \subset Z \) is open then \( g^{-1}(V) \) is open by continuity of \( g \).
- So we have open
  \[ g^{-1}(V) = (f \circ q)^{-1}(V) = q^{-1}(f^{-1}(V)) \]
- By def. of quotient top. \( f^{-1}(V) \) is open in \( X/\sim \).

Example 188 (Continuous functions on a quotient space)

- Let \( \sim \) on \( \mathbb{R} \) be the equiv. relation \( a \sim b \) if there is \( n \in \mathbb{Z} \) such that \( a = 2^n b \).
- Let \( W = \mathbb{R}/\sim \) and let \( q : \mathbb{R} \to W \) be the quotient map.
- What are the continuous functions \( f : W \to \mathbb{R} \)?
  - Let \( f : W \to \mathbb{R} \) be continuous.
  - Let \( g : \mathbb{R} \to \mathbb{R} \) be \( g = f \circ q \) which is continuous.
  - For any \( x \in \mathbb{R} - \{0\} \) and \( n \in \mathbb{Z} \) we have \( g(2^n x) = f \circ q(2^n x) = f([x]) \)
  - In \( \mathbb{R} \) the sequence \( 2^n x \to 0 \) so we must have
    \[ f([x]) = g(2^n x) \to g(0) = f([0]) \]
  - Hence for any \( x \in \mathbb{R} \) we have \( f([x]) = f([0]) \).
  - Hence \( f : W \to \mathbb{R} \) is continuous iff \( f \) is constant.
Quotient Topology

**Proposition 189 (Continuous functions on a graph)**

Let $X$ be a graph and $Z$ be a topological space. A function $f : X \to Z$ is continuous iff $f_\alpha : l_\alpha \to Z$ is continuous for every edge $l_\alpha$ of $X$ where $f_\alpha = f \circ q \circ i_\alpha$

**Proof.**

1. Let $Y = X^0 \amalg \bigcup_{\alpha \in A} l_\alpha$
2. $\varphi_\alpha : \partial l_\alpha \to X^0$ where $\partial l_\alpha = \{0,1\} \subset l_\alpha$
3. $i_\alpha : l_\alpha \to Y$ is the inclusion map.
4. and $\sim$ be as in Definition 186 (That is $\sim$ is the smallest equiv. rel. on $Y$ s.t. $y \sim \varphi_\alpha(y)$ for all $y \in \{0,1\} \subset l_\alpha$).
5. $q : Y \to X$ be the quotient map $q(y) = [y]$.
6. $f : X \to Z$ be a function with $f_\alpha = f \circ q \circ i_\alpha$ continuous for each $\alpha$.

Let $h : X^0 \to Z$ be the function $h = f \circ q$.

Proof of Prop. 189 (continued).

1. $X^0$ is has discrete top. so $h$ is continuous.
2. Then by universal prop. of coproduct top. we get continuous $g : Y \to Z$ with $g|_{l_\alpha} = f_\alpha$ and $g|_{X^0} = h$.
3. Suppose $a \in l_\alpha$ and $b \in l_\beta$ and $a \sim b$.
4. Then $g(a) = f_\alpha(a) = f \circ q \circ i_\alpha(a) = f \circ q \circ i_\beta(b) = f_\beta(b) = g(b)$.
5. Suppose $a \in l_\alpha$ and $b \in X^0$ and $a \sim b$.
6. Then $g(a) = f_\alpha(a) f \circ q \circ i_\alpha(a) = f \circ q(b) = h(b) = g(b)$.
7. Thus $g$ is constant on equiv. classes.
8. Hence by universal property of quotient topology $g$ induces the cont. function $f : X \to Z$. 

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Group Theory

Definition 190 (Group)
A group \( G \) is a set \( G \) with an operation \( \cdot : G \times G \to G \) such that
1. (Associativity) For all \( x, y, z \in G \) we have \((xy)z = x(yz)\)
2. (Identity) There is \( e \in G \) such that for all \( x \in G \) we have \( ex = x \)
3. (Inverses) For all \( x \in G \) there is \( y \in G \) such that \( xy = e \).

▶ The element \( e \in G \) is unique (prove this!) and is called the identity of \( G \) and is written 1.
▶ Given \( x \in G \) the element \( y \) with \( xy = 1 \) is unique and is called the inverse of \( x \) and is written \( x^{-1} \).

Definition 191 (Topological Group)
A topological group \( G \) is a set group for which \( \cdot : G \times G \to G \) is continuous.

Examples 192 (Topological Groups)
1. \((\mathbb{R}, +)\) is a topological group. YES since \(+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is cont.
2. Is \((\mathbb{R}, \cdot)\) a topological group? NO since 0 has no inverse in \((\mathbb{R}, \cdot)\)
3. \((\mathbb{R}^*, \cdot)\) is a topological group where \( \mathbb{R}^* = \mathbb{R} \setminus \{0\} \)
4. \((\mathbb{Z}, +)\) is a topological group. YES since \(+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}\) is cont.
5. Any group \( G \) with the discrete topology is a topological group.
6. \(\text{GL}_n(\mathbb{R}) \subset \text{Mat}_{n,n}(\mathbb{R}) = \mathbb{R}^{(n^2)}\) is a topological group.

Definition 193 (Subgroup)
A subgroup \( H \) of a group \( G \) is subset \( H \subset G \) satisfying:
1. \( 1 \in H \).
2. If \( x, y \in H \) then \( xy \in H \).
3. If \( x \in H \) then \( x^{-1} \in H \).
### Group Theory

**Definition 194 (Normal Subgroup)**

A **subgroup** $H$ of a group $G$ is **normal** if for all $g \in G$ and $h \in H$ we have $ghg^{-1} \in H$.

**Definition 195 (Homomorphism)**

A **homomorphism** from a group $G$ to a group $H$ is a function $h : G \to H$ satisfying

1. $h(1_G) = 1_H$
2. For all $x, y \in G$ we have $h(xy) = h(x) \cdot h(y)$.

**Proposition 196**

*If $H$ of the topological group $G$ then the closure $\overline{H}$ of $H$ is also a subgroup of $G$.*

### Category Theory

**Definition 197 (Category)**

A **category** $\mathcal{C}$ consists of

1. **objects** $\text{Ob}(\mathcal{C})$ of $\mathcal{C}$.
2. For all $X, Y \in \text{Ob}(\mathcal{C})$ a set $\text{Mor}(X, Y)$ called **morphisms** from $X$ to $Y$.
3. For all $X, Y, Z \in \text{Ob}(\mathcal{C})$ a “composition”
   $\circ : \text{Mor}(X, Y) \times \text{Mor}(Y, Z) \to \text{Mor}(X, Z)$

Satisfying

1. For all $X \in \text{Ob}(\mathcal{C})$ there is distinguished $\text{id} = \text{id}_X \in \text{Mor}(X, X)$ s.t.
   $f \circ \text{id} = \text{id} \circ f = f$ for all morphisms $f$.
2. For all morphisms $f, g, h$ we have $f \circ (g \circ h) = (f \circ g) \circ h$.
Category Theory

Example 198 (Set)

\[ \mathcal{C} = \text{Set} \]
- \( \text{Ob}(\text{Set}) \) are sets.
- \( \text{Mor}(X, Y) \) are functions \( f : X \to Y \).
- \( \text{id} \in \text{Mor}(X, X) \) is identity function \( (\text{id}_X : X \to X) \).

Example 199 (Top)

\[ \mathcal{C} = \text{Top} \]
- \( \text{Ob}(\text{Top}) \) is all groups
- \( \text{Mor}(X, Y) \) continuous maps from \( X \) to \( Y \) with composition.
- \( \text{id} \in \text{Mor}(X, X) \) is identity function \( (\text{id}_X : X \to X) \).

Example 200 (Group)

\[ \mathcal{C} = \text{Group} \]
- \( \text{Ob}(\text{Group}) \) is all groups
- \( \text{Mor}(G, H) \) homomorphisms from \( G \) to \( H \) with composition.
- \( \text{id} \in \text{Mor}(G, G) \) is identity function \( (\text{id}_G : G \to G) \).
Definition 201 (Functor)

A functor $F$ from a category $C$ to a category $D$ consists of

1. For each object $X \in \text{Ob}(C)$ an object $F(X) \in \text{Ob}(D)$
2. For each morphism $f : X \rightarrow Y$ in $\text{Mor}(C)$ a morphism $F(f) : F(X) \rightarrow F(Y)$ in $\text{Mor}(D)$

Satisfying

1. For all $X \in \text{Ob}(C)$ we have $F(\text{id}_X) = \text{id}_{F(X)}$
2. $F(f \circ g) = F(f) \circ F(g)$