Math 5801
General Topology and Knot Theory

Nathan Broaddus

Ohio State University

October 10, 2012

Course Info

Reading for Friday, October 12

Chapter 3.24, pgs. 153-157

HW 7 for Monday, October 15

- Chapter 2.21: 7, 8, 12b
- Chapter 2.22: 2a-b, 4a, 5
- Chapter 3.23: 2, 4, 11

Quotient Topology

- Let X be a topological space and $\sim$ be an equivalence relation on X.
- Let Z be a topological space.
- What are the continuous functions $f : (X/\sim) \to Z$?
- We have that the quotient map $q : X \to X/\sim$ is continuous so any continuous function $f : (X/\sim) \to Z$ gives continuous $f \circ q : X \to Z$.
- For which functions $g : X \to Z$ is there a function $f : (X/\sim) \to Z$ such that $g = f \circ q$?

Proposition 187 (Universal Property of Quotient Space)

1. *Let X be a topological space and $\sim$ be an equivalence relation on X with quotient map $q : X \to X/\sim$*
2. *Let Z be a topological space.*
3. *Let $g : X \to Z$ be a continuous function.*

There is a continuous function $f : (X/\sim) \to Z$ such that $g = f \circ q$ iff for each $[x] \in X/\sim$ we have that $g|_{q^{-1}}(\{[x]\})$ is constant.

Quotient Topology

Proof of Prop. 187.

- Let X be a topological space and $\sim$ be an equivalence relation on X with quotient map $q : X \to X/\sim$
- Let Z be a topological space.
- Let $g : X \to Z$ be a continuous function.
- $\Rightarrow$
  - Suppose there is continuous $f : (X/\sim) \to Z$ such that $g = f \circ q$.
  - If $x_0, x_1 \in X$ and $x_0 \sim x_1$ then

$$g(x_0) = f \circ q(x_0) = f([x_0]) = f([x_1]) = f \circ q(x_1) = g(x_1)$$

Quotient Topology

Proof of Prop. 187 *(continued)*.

- $\Leftarrow$
  - Suppose that for all $x_0, x_1 \in X$ if $x_0 \sim x_1$ then $g(x_0) = g(x_1)$
  - Define $f : X/\sim \to Z$ to be the function $f([x]) = g(x)$.
  - By assumption f is well-defined.
  - If $V \subset Z$ is open then $g^{-1}(V)$ is open by continuity of g.
  - So we have open

$$g^{-1}(V) = (f \circ q)^{-1}(V) = q^{-1}(f^{-1}(V))$$

  - By def. of quotient top. $f^{-1}(V)$ is open in $X/\sim$.

$\square$

Quotient Topology

Example 188 (Continuous functions on a quotient space)

- Let $\sim$ on $\mathbf{R}$ be the equiv. relation $a \sim b$ if there is $n \in \mathbf{Z}$ such that $a = 2^n b$.
- Let $W = \mathbf{R}/\sim$ and let $q : \mathbf{R} \to W$ be the quotient map.
- What are the continuous functions $f : W \to \mathbf{R}$?
  - Let $f : W \to \mathbf{R}$ be continuous.
  - Let $g : \mathbf{R} \to \mathbf{R}$ be $g = f \circ q$ which is continuous.
  - For any $x \in \mathbf{R} - \{0\}$ and $n \in \mathbf{Z}$ we have $g(2^n x) = f \circ q(2^n x) = f([x])$
  - In $\mathbf{R}$ the sequence $2^n x \to 0$ so we must have

$$f([x]) = g(2^n x) \to g(0) = f([0])$$

  - Hence for any $x \in \mathbf{R}$ we have $f([x]) = f([0])$.
- Hence $f : W \to \mathbf{R}$ is continuous iff f is constant.

Quotient Topology

Proposition 189 (Continuous functions on a graph)

Let X be a graph and Z be a topological space. A function $f : X \to Z$ is continuous iff $f_\alpha : I_\alpha \to Z$ is continuous for every edge I_α of X where $f_\alpha = f \circ q \circ i_\alpha$

Proof.

- Let
  1. Let $Y = X^0 \amalg \coprod_{\alpha \in \mathcal{A}} I_\alpha$
  2. $\varphi_\alpha : \partial I_\alpha \to X^0$ where $\partial I_\alpha = \{0, 1\} \subset I_\alpha$
  3. $i_\alpha : I_\alpha \to Y$ is the inclusion map.
  4. and $\sim$ be as in Definition 186 (That is $\sim$ is the smallest equiv. rel. on Y s.t. $y \sim \varphi_\alpha(y)$ for all $y \in \{0, 1\} \subset I_\alpha$).
  5. $q : Y \to X$ be the quotient map $q(y) = [y]$.
  6. $f : X \to Z$ is a function with $f_\alpha = f \circ q \circ i_\alpha$ continuous for each α.
- Let $h : X^0 \to Z$ be the function $h = f \circ q$.

Quotient Topology

Proof of Prop. 189 *(continued)*.

- X^0 is has discrete top. so h is continuous.
- Then by universal prop. of coproduct top. we get continuous $g : Y \to Z$ with $g|_{I\alpha} = f_\alpha$ and $g|_{X^0} = h$.
- Suppose $a \in I_\alpha$ and $b \in I_\beta$ and $a \sim b$.
- Then $g(a) = f_\alpha(a) = f \circ q \circ i_\alpha(a) = f \circ q \circ i_\beta(b) = f_\beta(b) = g(b)$.
- Suppose $a \in I_\alpha$ and $b \in X^0$ and $a \sim b$.
- Then $g(a) = f_\alpha(a) f \circ q \circ i_\alpha(a) = f \circ q(b) = h(b) = g(b)$.
- Thus g is constant on equiv. classes.
- Hence by universal property of quotient topology g induces the cont. function $f : X \to Z$.

$\square$

Group Theory

Definition 190 (Group)

A **group** G is a set G with an operation $\cdot : G \times G \to G$ such that

1. *(Associativity)* For all $x, y, z \in G$ we have $(xy)z = x(yz)$
2. *(Identity)* There is $e \in G$ such that for all $x \in G$ we have $ex = x$
3. *(Inverses)* For all $x \in G$ there is $y \in G$ such that $xy = e$.

- The element $e \in G$ is unique (prove this!) and is called the **identity** of G and is written 1.
- Given $x \in G$ the element y with $xy = 1$ is unique and is called the **inverse** of x and is written x^{-1}.

Definition 191 (Topological Group)

A **topological group** G is a set group for which $\cdot : G \times G \to G$ is continuous.

Group Theory

Examples 192 (Topological Groups)

1. $(\mathbf{R}, +)$ is a topological group. YES since $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is cont.
2. Is $(\mathbf{R}, \cdot)$ is a topological group? NO since 0 has no inverse in $(\mathbf{R}, \cdot)$
3. $(\mathbf{R}^*, \cdot)$ is a topological group where $\mathbf{R}^* = \mathbf{R} - \{0\}$
4. $(\mathbf{Z}, +)$ is a topological group. YES since $+ : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is cont.
5. Any group G with the discrete topology is a topological group.
6. $\mathrm{GL}_n(\mathbf{R}) \subset \mathrm{Mat}_{n,n}(\mathbf{R}) = \mathbf{R}^{(n^2)}$ is a topological group.

Definition 193 (Subroup)

A **subgroup** H of a group G is subset $H \subset G$ satisfying:

1. $1 \in H$.
2. If $x, y \in H$ then $xy \in H$.
3. If $x \in H$ then $x^{-1} \in H$.

Group Theory

Definition 194 (Normal Subroup)

A **subgroup** H of a group G is **normal** if for all $g \in G$ and $h \in H$ we have $ghg^{-1} \in H$.

Definition 195 (Homomorphism)

A **homomorphism** from a group G to a group H is a function $h : G \to H$ satisfying

1. $h(1_G) = 1_H$
2. For all $x, y \in G$ we have $h(xy) = h(x) \cdot h(y)$.

Proposition 196

If H of the topological group G then the closure $\overline{H}$ of H is also a subgroup of G.

Category Theory

Definition 197 (Category)

A **category** $\mathcal{C}$ consists of

1. **objects** $\mathrm{Ob}(\mathcal{C})$ of $\mathcal{C}$.
2. For all $X, Y \in \mathrm{Ob}(\mathcal{C})$ a set $\mathrm{Mor}(X, Y)$ called **morphisms** from X to Y
3. For all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a "composition" $\circ : \mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z)$

Satisfying

1. For all $X \in \mathrm{Ob}(\mathcal{C})$ there is distinguished $\mathrm{id} = \mathrm{id}_X \in \mathrm{Mor}(X, X)$ s.t. $f \circ \mathrm{id} = \mathrm{id} \circ f = f$ for all morphisms f.
2. For all morphisms f, g, h we have $f \circ (g \circ h) = (f \circ g) \circ h$

Category Theory

Example 198 (**Set**)

$\mathcal{C} =$ **Set**

- Ob(**Set**) are sets.
- Mor(X, Y) are functions $f : X \to Y$.
- $\mathrm{id} \in$ Mor(X, X) is identity function ($\mathrm{id}_X : X \to X$).

Example 199 (**Top**)

$\mathcal{C} =$ **Top**

- Ob(**Top**) is all groups
- Mor(X, Y) continuous maps from X to Y with composition.
- $\mathrm{id} \in$ Mor(X, X) is identity function ($\mathrm{id}_X : X \to X$).

Category Theory

Example 200 (**Group**)

$\mathcal{C} =$ **Group**

- Ob(**Group**) is all groups
- Mor(G, H) homomorphisms from G to H with composition.
- $\mathrm{id} \in$ Mor(G, G) is identity function ($\mathrm{id}_G : G \to G$).

Where we are

Where we are

- We've introduced basic top. spaces $\mathbf{R}$, $\mathbf{R}^n$, $\mathbf{R}^\omega$, $\mathbf{Z}_f$, *etc.*
- Constructions of top. spaces:
- Order topology
- Subspace topology
- Product topology
- Quotient topology

Where we're headed

- $\mathbf{R}$ is a very nice top. space
- We'd like to identify some of it's nice properties
- We've already talked about it's metric
- Now we'll talk about its *connectedness*
- Later *compactness* of subsets of form $[a, b]$

$\sqrt{2}$

Why does $\sqrt{2}$ exist?

- Why is there a real number r s.t. $r^2 = 2$?
- Intermediate Value Theorem.

Connectedness

Definition 201

Separation and connectedness A **separation** of a topological space X is two disjoint open sets $U, V \subset X$ such that $X = U \cup V$.
A space is **connected** if it has no separation.

Examples 202

1. Let $X = (1, 3] \cup [8, 10] \subset \mathbf{R}$ with the subset topology. X has a separation $U = (1, 3], V = [8, 10]$
2. Any space X with the trivial topology is connected.
3. In a connected space X a nonempty set which is open and closed must be X.
4. $\mathbf{Z}$ is not connected.
5. $\mathbf{Q}$ is not connected.