Course Info

Reading for Friday, October 12
Chapter 3.24, pgs. 153-157

HW 7 for Monday, October 15
- Chapter 2.21: 7, 8, 12b
- Chapter 2.22: 2a-b, 4a, 5
- Chapter 3.23: 2, 4, 11
Quotient Topology

Let $X$ be a topological space and $\sim$ be an equivalence relation on $X$.

Let $Z$ be a topological space.

What are the continuous functions $f : (X/\sim) \to Z$?

We have that the quotient map $q : X \to X/\sim$ is continuous so any continuous function $f : (X/\sim) \to Z$ gives continuous $f \circ q : X \to Z$.

For which functions $g : X \to Z$ is there a function $f : (X/\sim) \to Z$ such that $g = f \circ q$?

### Proposition 187 (Universal Property of Quotient Space)

1. Let $X$ be a topological space and $\sim$ be an equivalence relation on $X$ with quotient map $q : X \to X/\sim$.
2. Let $Z$ be a topological space.
3. Let $g : X \to Z$ be a continuous function.

There is a continuous function $f : (X/\sim) \to Z$ such that $g = f \circ q$ iff for each $[x] \in X/\sim$ we have that $g \big|_{q^{-1}([x])}$ is constant.

### Proof of Prop. 187.

Let $X$ be a topological space and $\sim$ be an equivalence relation on $X$ with quotient map $q : X \to X/\sim$.

Let $Z$ be a topological space.

Let $g : X \to Z$ be a continuous function.

Suppose there is continuous $f : (X/\sim) \to Z$ such that $g = f \circ q$.

If $x_0, x_1 \in X$ and $x_0 \sim x_1$ then

$$g(x_0) = f \circ q(x_0) = f([x_0]) = f([x_1]) = f \circ q(x_1) = g(x_1)$$
Proof of Prop. 187 (continued).

$\Leftarrow$

$\Rightarrow$ Suppose that for all $x_0, x_1 \in X$ if $x_0 \sim x_1$ then $g(x_0) = g(x_1)$

$\Rightarrow$ Define $f : X/\sim \rightarrow Z$ to be the function $f([x]) = g(x)$.

$\Rightarrow$ By assumption $f$ is well-defined.

$\Rightarrow$ If $V \subset Z$ is open then $g^{-1}(V)$ is open by continuity of $g$.

$\Rightarrow$ So we have open

$$g^{-1}(V) = (f \circ q)^{-1}(V) = q^{-1}(f^{-1}(V))$$

$\Rightarrow$ By def. of quotient top. $f^{-1}(V)$ is open in $X/\sim$.

Example 188 (Continuous functions on a quotient space)

$\Rightarrow$ Let $\sim$ on $\mathbb{R}$ be the equiv. relation $a \sim b$ if there is $n \in \mathbb{Z}$ such that $a = 2^n b$.

$\Rightarrow$ Let $W = \mathbb{R}/\sim$ and let $q : \mathbb{R} \rightarrow W$ be the quotient map.

$\Rightarrow$ What are the continuous functions $f : W \rightarrow \mathbb{R}$?

$\Rightarrow$ Let $f : W \rightarrow \mathbb{R}$ be continuous.

$\Rightarrow$ Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be $g = f \circ q$ which is continuous.

$\Rightarrow$ For any $x \in \mathbb{R} - \{0\}$ and $n \in \mathbb{Z}$ we have $g(2^n x) = f \circ q(2^n x) = f([x])$

$\Rightarrow$ In $\mathbb{R}$ the sequence $2^n x \rightarrow 0$ so we must have

$$f([x]) = g(2^n x) \rightarrow g(0) = f([0])$$

$\Rightarrow$ Hence for any $x \in \mathbb{R}$ we have $f([x]) = f([0])$.

$\Rightarrow$ Hence $f : W \rightarrow \mathbb{R}$ is continuous iff $f$ is constant.
Quotient Topology

Proposition 189 (Continuous functions on a graph)

Let $X$ be a graph and $Z$ be a topological space. A function $f : X \rightarrow Z$ is continuous iff $f_\alpha : i_\alpha \rightarrow Z$ is continuous for every edge $I_\alpha$ of $X$ where $f_\alpha = f \circ q \circ i_\alpha$.

Proof.

1. Let $Y = X^0 \amalg \bigoplus_{\alpha \in A} I_\alpha$
2. $\varphi : \partial I_\alpha \rightarrow X^0$ where $\partial I_\alpha = \{0, 1\} \subset I_\alpha$
3. $i_\alpha : I_\alpha \rightarrow Y$ is the inclusion map.
4. and $\sim$ be as in Definition 186 (That is $\sim$ is the smallest equiv. rel. on $Y$ s.t. $y \sim \varphi(y)$ for all $y \in \{0, 1\} \subset I_\alpha$).
5. $q : Y \rightarrow X$ be the quotient map $q(y) = [y]$. 
6. $f : X \rightarrow Z$ be a function with $f_\alpha = f \circ q \circ i_\alpha$ continuous for each $\alpha$.

Let $h : X^0 \rightarrow Z$ be the function $h = f \circ q$.

Quotient Topology

Proof of Prop. 189 (continued).

$X^0$ is has discrete top. so $h$ is continuous.

Then by universal prop. of coproduct top. we get continuous $g : Y \rightarrow Z$ with $g|_{I_\alpha} = f_\alpha$ and $g|_{X^0} = h$.

Suppose $a \in I_\alpha$ and $b \in I_\beta$ and $a \sim b$.
Then $g(a) = f_\alpha(a) = f \circ q \circ i_\alpha(a) = f \circ q \circ i_\beta(b) = g(b)$.

Suppose $a \in I_\alpha$ and $b \in X^0$ and $a \sim b$.
Then $g(a) = f_\alpha(a) f \circ q \circ i_\alpha(a) = f \circ q(b) = h(b) = g(b)$.

Thus $g$ is constant on equiv. classes.

Hence by universal property of quotient topology $g$ induces the cont. function $f : X \rightarrow Z$.
Definition 190 (Group)
A group $G$ is a set $G$ with an operation $\cdot : G \times G \to G$ such that
1. (Associativity) For all $x, y, z \in G$ we have $(xy)z = x(yz)$
2. (Identity) There is $e \in G$ such that for all $x \in G$ we have $ex = x$
3. (Inverses) For all $x \in G$ there is $y \in G$ such that $xy = e$.
   ▶ The element $e \in G$ is unique (prove this!) and is called the identity of $G$ and is written $1$.
   ▶ Given $x \in G$ the element $y$ with $xy = 1$ is unique and is called the inverse of $x$ and is written $x^{-1}$.

Definition 191 (Topological Group)
A topological group $G$ is a set group for which $\cdot : G \times G \to G$ is continuous.

Examples 192 (Topological Groups)
1. $(\mathbb{R}, +)$ is a topological group. YES since $+: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is cont.
2. Is $(\mathbb{R}, \cdot)$ a topological group? NO since $0$ has no inverse in $(\mathbb{R}, \cdot)$
3. $(\mathbb{R}^*, \cdot)$ is a topological group where $\mathbb{R}^* = \mathbb{R} - \{0\}$
4. $(\mathbb{Z}, +)$ is a topological group. YES since $+: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is cont.
5. Any group $G$ with the discrete topology is a topological group.
6. $\text{GL}_n(\mathbb{R}) \subset \text{Mat}_{n \times n}(\mathbb{R}) = \mathbb{R}^{(n^2)}$ is a topological group.

Definition 193 (Subgroup)
A subgroup $H$ of a group $G$ is subset $H \subset G$ satisfying:
1. $1 \in H$.
2. If $x, y \in H$ then $xy \in H$.
3. If $x \in H$ then $x^{-1} \in H$. 
### Group Theory

**Definition 194 (Normal Subgroup)**

A **subgroup** $H$ of a group $G$ is **normal** if for all $g \in G$ and $h \in H$ we have $ghg^{-1} \in H$.

**Definition 195 (Homomorphism)**

A **homomorphism** from a group $G$ to a group $H$ is a function $h : G \rightarrow H$ satisfying

1. $h(1_G) = 1_H$
2. For all $x, y \in G$ we have $h(xy) = h(x) \cdot h(y)$.

**Proposition 196**

If $H$ of the topological group $G$ then the closure $\overline{H}$ of $H$ is also a subgroup of $G$.

### Category Theory

**Definition 197 (Category)**

A **category** $\mathcal{C}$ consists of

1. **objects** $\text{Ob}(\mathcal{C})$ of $\mathcal{C}$.
2. For all $X, Y \in \text{Ob}(\mathcal{C})$ a set $\text{Mor}(X, Y)$ called **morphisms** from $X$ to $Y$.
3. For all $X, Y, Z \in \text{Ob}(\mathcal{C})$ a “composition” $\circ : \text{Mor}(X, Y) \times \text{Mor}(Y, Z) \rightarrow \text{Mor}(X, Z)$

Satisfying

1. For all $X \in \text{Ob}(\mathcal{C})$ there is distinguished $\text{id} = \text{id}_X \in \text{Mor}(X, X)$ s.t. $f \circ \text{id} = \text{id} \circ f = f$ for all morphisms $f$.
2. For all morphisms $f, g, h$ we have $f \circ (g \circ h) = (f \circ g) \circ h$.
### Category Theory

#### Example 198 (Set)

\[ C = \text{Set} \]
- \( \text{Ob}(\text{Set}) \) are sets.
- \( \text{Mor}(X, Y) \) are functions \( f : X \to Y \).
- \( \text{id} \in \text{Mor}(X, X) \) is identity function \( \text{id}_X : X \to X \).

#### Example 199 (Top)

\[ C = \text{Top} \]
- \( \text{Ob}(\text{Top}) \) is all groups
- \( \text{Mor}(X, Y) \) continuous maps from \( X \) to \( Y \) with composition.
- \( \text{id} \in \text{Mor}(X, X) \) is identity function \( \text{id}_X : X \to X \).

#### Example 200 (Group)

\[ C = \text{Group} \]
- \( \text{Ob}(\text{Group}) \) is all groups
- \( \text{Mor}(G, H) \) homomorphisms from \( G \) to \( H \) with composition.
- \( \text{id} \in \text{Mor}(G, G) \) is identity function \( \text{id}_G : G \to G \).
Where we are

We’ve introduced basic top. spaces $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^\omega, \mathbb{Z}_f$, etc.
- Constructions of top. spaces:
  - Order topology
  - Subspace topology
  - Product topology
  - Quotient topology

Where we’re headed

- $\mathbb{R}$ is a very nice top. space
- We’d like to identify some of it’s nice properties
- We’ve already talked about it’s metric
- Now we’ll talk about its connectedness
- Later compactness of subsets of form $[a, b]$

Why does $\sqrt{2}$ exist?

- Why is there a real number $r$ s.t. $r^2 = 2$?
- Intermediate Value Theorem.
**Definition 201**

Separation and connectedness A *separation* of a topological space $X$ is two disjoint open sets $U, V \subset X$ such that $X = U \cup V$. A space is **connected** if it has no separation.

**Examples 202**

1. Let $X = (1, 3] \cup [8, 10] \subset \mathbb{R}$ with the subset topology. $X$ has a separation $U = (1, 3]$, $V = [8, 10]$.
2. Any space $X$ with the trivial topology is connected.
3. In a connected space $X$ a nonempty set which is open and closed must be $X$.
4. $\mathbb{Z}$ is not connected.
5. $\mathbb{Q}$ is not connected.