Math 5801
General Topology and Knot Theory

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Course Info

Reading for Wednesday, October 17

Review Chapter 3.25

HW 8 for Monday, October 22

- Chapter 2.24: 3, 5a-d, 8a-d, 12a-f (see pg. 66 for required definitions)
- Chapter 2.25: 1, 2a-c

Connectedness

Proposition 208 (**R** is connected)

R *is connected.*

Proof.

- Suppose A and B give a separation of **R**.
- Then there is $a \in A$ and $b \in B$. WLOG assume $a < b$.
- Let $A_0 = A \cap [a, b]$ and $B_0 = B \cap [a, b]$.
- A_0 is nonempty and bounded so there is $\alpha \in [a, b]$ s.t. $c = \sup A_0$.
- CASE I: Suppose $c \in B_0$
  - If $c \neq a$ since $c \in B_0$ and $a \in A_0$
  - b has a nbdh $(d, b] \subset B_0$ so $c \neq b$.
  - If $a < c < b$ then $(c, b] \subset B_0$ and c has a nbdh $(d, e) \subset B_0$ so d is an upper bound of A_0 contradicting $c = \sup A_0$.

Connectedness

Proof of Prop. 208 *(continued).*

- CASE II: Suppose $c \in A_0$
  - If $c \neq b$ since $c \in A_0$ and $b \in B_0$
  - a has a nbdh $[a, d) \subset A_0$ so $c \neq a$.
  - If $a < c < b$ then there is $[c, d) \subset A_0$ so c is not an upper bound of A_0 contradicting $c = \sup A_0$.

□

Corollary 209

Interval subsets of **R** *are connected*

Connectedness

Definition 210 (Linear continuum)

A **linear continuum** is an ordered set X satisfying

1. X has satisfies the (LUB) axiom
2. For all $x, z \in X$ with $x < z$ there is $y \in X$ with $x < y < z$.

X is given the order topology.

- Actually proof of Prop. 208 shows any linear continuum is connected.

Corollary 211

Interval subsets of $\mathbf{R}$ *are connected.*

Connectedness

Theorem 212 (Intermediate Value Theorem)

Let X be a connected space and Y be an ordered set with the order topology. If there is a continuous funtion $f : X \to Y$ and $y \in Y$ such that there are $a, b \in X$ with $f(a) < y < f(b)$ then there is $x \in X$ such that $f(x) = y$.

Proof.

- Suppose there is no $x \in X$ such that $f(x) = y$.
- The sets $f^{-1}((-\infty, y))$ and $f^{-1}((y, \infty))$ are disjoint open and nonempty.
- Hence they give a separation of X.
- Thus X is not connected.

□

Connectedness

Definition 213 (Path)

A **path** in a space X is a continuous function $f : [0,1] \to X$. The **endpoints** of the path $f : [0,1] \to X$ are the points $f(0), f(1) \in X$.

Definition 214 (Path connected)

A space X is **path connected** if for all $x, y \in X$ there is a path $f : [0,1] \to X$ with endpoints x and y.

Examples 215 (Path connected spaces)

1. $\mathbf{R}$ is path connected since $f(t) = a(1-t) + bt$ is a path from a to b.
2. $\mathbf{R}^J$ is path connected since if $f_\alpha(t) = a_\alpha(1-t) + b_\alpha t$ and we set $f : [0,1] \to \mathbf{R}^J$ to satisfy $f(t) = (f_\alpha(t))_{\alpha \in J}$ then f is a path from $(a_\alpha)_{\alpha \in J}$ to $(b_\alpha)_{\alpha \in J}$.

Connectedness

Proposition 216 (Continuous images of path connected spaces are path connected)

If X is a path connected and $f : X \to Y$ is continuous then $f(X)$ is a path connected subspace of Y.

Proof.

- Let $y_0, y_1 \in f(X)$.
- There are $x_0, x_1 \in X$ such that $f(x_0) = y_0$ and $f(x_1) = y_1$
- Let $p : [0,1] \to X$ be a path from x_0 to x_1.
- Then $f \circ p$ is a path from y_0 to y_1.

□

Connectedness

- How does path connectedness relate to connectedness?

Proposition 217 (Path connected implies connected)

If X is a path connected space then X is connected.

Proof.

- Fix $x_0 \in X$
- for each $x \in X$ let $f_x : [0,1] \to X$ be a path from x_0 to x.
- $X = \bigcup_{x \in X} f_x([0,1])$ is a union of connected sets.

□

- What about the other direction?

Connectedness

Proposition 218 (Connected does not imply path connected)

There is a space which is connected but not path connected.

Example 219 (Ordered square is connected but not path connected)

- Let $I_o^2 = I \times I$ with dictionary order topology.
- I_o^2 is a linear continuum and hence connected.
- Suppose $f : [0,1] \to I_o^2$ is continuous and satisfies $f(0) = (0,0)$ and $f(1) = (1,1)$
- For each $r \in [0,1]$ the set $\{r\} \times (\frac{1}{3}, \frac{2}{3})$ is open so $f^{-1}(\{r\} \times (\frac{1}{3}, \frac{2}{3}))$ is open in $[0,1]$.
- Thus it contains some $q \in \mathbf{Q} \cap [0,1]$.
- Thus $\pi_1 \circ f$ is a surjection from countable $\mathbf{Q} \cap [0,1]$ to uncountable $[0,1]$.
- Contradiction.

Connectedness

Example 220 (Topologist's sine curve)

- Let $X \subset \mathbf{R}^2$ be the following subspace

$$U = \{(x, \sin(\tfrac{1}{x}))|x > 0\}$$
$$V = \{(0, y)|0 \leq y \leq 1\}$$
$$X = U \cup V$$

- U is path connected and hence connected.
- $U \subset X \subset \overline{U}$ so X is connected.
- No path can connect $(0, y)$ to $(1, \sin 1)$.
  - Suppose $f : [0, 1] \to X$ is a path from $(0, y)$ to $(1, \sin 1)$ with $f(t) = (x(t), y(t))$.
  - Assume $x(0) = 0$ and $x(t) > 0$ for $t > 0$ otherwise restrict f to smaller domain.
  - Choose $(t_n)_{n \in \mathbf{Z}_+}$ to be a sequence in I with $y(t_n) = (-1)^n$ and $t_n \to 0$
  - f cannot be continuous.

Components

Definition 221 (Components of a space)

Let X be topological space. For each $x, y \in X$ set $x \sim y$ if X has a connected subset A with $x, y \in A$. A **component** of X is an equivalence class of $\sim$.

- Why is $\sim$ transitive?

Proposition 222 (Connected does not imply path connected)

Components of X are connected and their union is X.