Math 5801
General Topology and Knot Theory

Nathan Broaddus

Ohio State University

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Course Info

Reading for Monday, October 22

Chapter 3.27, pgs. 172-177

HW 8 for Monday, October 22

- Chapter 2.24: 3, 5a-d, 8a-d, 12a-f (see pg. 66 for required definitions)
- Chapter 2.25: 1, 2a-c

Components

Proposition 229

A space X is locally connected if and only if for every open subset $U \subset X$ each component of U is open in X.

Proof.

- Suppose X is locally connected.
  - Let $U \subset X$ be open.
  - Let $C \subset U$ be a component of U.
  - Let $x \in C$.
  - x has a connected open nbdh $V_x \subset U$.
  - By definition of component $V_x \subset C$.
  - Thus $C = \bigcup_{x \in C} V_x$ is open.
- Suppose for every open subset $U \subset X$ each component of U is open in X.
  - Let $U \subset X$ be open.
  - Let $C \subset U$ be the component of $x \in U$.
  - C is a connected open neighborhood of x contained in U. □

Components

Proposition 230

A space X is locally path connected if and only if for every open subset $U \subset X$ each path component of U is open in X.

Proposition 231

If X is locally path connected then its components are also path components.

Compactness

- Intermediate Value Theorem for $\mathbf{R}$ lead us to notion of *connectedess.*
- Another classic theorem for continuous functions on $\mathbf{R}$ is the Maximum Theorem:

$$\textit{If } f : [a, b] \to \mathbf{R} \textit{ is continuous then there is } y \in [a, b] \textit{ s.t. for all } x \in [a, b] \textit{ we have } f(y) \geq f(x).$$

- What topological property of $[a, b]$ ensures that continuous images achieve a maximum value?
- We want a property P such that if X has property P then for any continuous $f : X \to \mathbf{R}$ there is $y \in X$ such that $\forall x \in X, f(y) \geq f(x)$.
- Note that this property is not shared by $\mathbf{R}$ or (a, b).
- On the other hand it should (probably) be shared by any finite set with the discrete topology.

Compactness

Definition 232 (Cover)

A **cover** of a topological space X is a collection $\mathcal{A}$ of subsets of X such that $\bigcup \mathcal{A} = X$.

Definition 233 (Subcover)

A **subcover** of a cover $\mathcal{A}$ of a topological space X is a collection $\mathcal{S} \subset \mathcal{A}$ such that $\bigcup \mathcal{S} = X$.

Definition 234 (Open cover)

An **open cover** of a topological space X is a collection $\mathcal{U}$ of open sets such that $\bigcup \mathcal{U} = X$.

Definition 235 (Compact)

A topological space X is **compact** if every open cover of X has a *finite* subcover.

Compactness

Examples 236 (Noncompact Spaces)

Showing that X is *not* compact only requires an infinite open cover of X which has no finite subcover:

1. $\mathbf{R}$ is not compact.
   - For each $n \in \mathbf{Z}$ let $U_n = (n, n + \frac{4}{3})$.
   - Let $\mathcal{U} = \{U_n | n \in \mathbf{Z}\}$.
   - $\mathcal{U}$ is an open cover of $\mathbf{R}$ since $\bigcup \mathcal{U} = \mathbf{R}$.
   - Let $\mathcal{S} \subset \mathcal{U}$ be a subcover.
   - For all $n \in \mathbf{Z}$ $n + 1 \in U_n$ and $n + 1 \notin U_m$ for $m \neq n$
   - Hence for each $n \in \mathbf{Z}$ we have $U_n \in \mathcal{S}$.
   - Thus the only subcover of $\mathcal{U}$ is $\mathcal{U}$ which is infinite.
2. $(0, 1)$ is not compact.
   - For each $n \in \mathbf{Z}_+$ let $U_n = (\frac{1}{n}, 1)$.
   - Let $\mathcal{U} = \{U_n | n \in \mathbf{Z}_+\}$.
3. $\mathbf{Q}$ and $\mathbf{Z}$ are not compact.
4. $\mathbf{R}^n$ is not compact.

Compactness

Examples 237 (Compact Spaces)

Showing that X is *not* compact only requires an infinite open cover of X which has no finite subcover:

1. $\mathbf{R}_f$ is compact.
   - Suppose $\mathcal{U}$ is an open cover of $\mathbf{R}_f$
   - There is nonempty $U_0 \in \mathcal{U}$.
   - There are finitely many $r_1, \cdots, r_n \in \mathbf{R}_f$ s.t. $r_1, \cdots, r_n \notin U_0$
   - $\mathcal{U}$ is a cover so there are $U_1, U_2, \cdots, U_n \in \mathcal{U}$ s.t. $r_1 \in U_1, \cdots, r_n \in U_n$
   - $U_0, U_1, \cdots, U_n$ is a finite subcover of $\mathcal{U}$
2. $K = \{0\} \cup \{\frac{1}{n} | n \in \mathbf{Z}_+\}$ is compact
   - Suppose $\mathcal{U}$ is an open cover of K.
   - Then there is $U_0 \in \mathcal{U}$ s.t. $0 \in U_0$
   - There is $N \in \mathbf{Z}_+$ s.t. for $n > N$ U_0contains $\frac{1}{n}$
   - $\mathcal{U}$ is a cover so there are $U_1, U_2, \cdots, U_n \in \mathcal{U}$ s.t. $\frac{1}{1} \in U_1, \cdots, \frac{1}{N} \in U_N$
   - $U_0, U_1, \cdots, U_N$ is a finite subcover of $\mathcal{U}$

Compactness

Definition 238 (Cover of a subspace)

If A is a subspace of X a collection of set $\mathcal{S}$ **covers** A if $A \subset \bigcup \mathcal{S}$.

Proposition 239

A subspace $A \subset X$ is compact if and only if any covering of A by open sets of X has a finite subcovering.

Compactness

- What properties do compact sets have?

Proposition 240 (Compact subsets of Hausdorff spaces are closed)

If X is Hausdorff and $K \subset X$ is compact then K is closed.

Proof.

- Suppose X is Hausdorff and $K \subset X$ is compact
- Fix $x \in X - K$.
- For each $k \in K$ let U_k be a nbdh of k disjoint from nbdh V_k of x.
- $\mathcal{U} = \{U_k | k \in K\}$ is an open cover of K
- $\mathcal{U}$ has a finite subcover $\{U_{k_1}, \cdots, U_{k_n}\}$
- So $V_x = \bigcap_{i=1}^{n} V_{k_i}$ is an open nbhd of x disjoint from K.
- So $X - K = \bigcup_{x \in X-K} V_x$ is open. $\square$

Compactness

- Tools for proving compactness

Proposition 241 (Continuous images of compact spaces are compact)

If X and Y are spaces and X is compact and $f : X \to Y$ is continuous then $f(X)$ is a compact subspace of Y.

Proof.

- Suppose X is compact and $f : X \to Y$ is continuous
- Let $\mathcal{V}$ be a cover of $f(X)$ by open sets in Y.
- Let $\mathcal{U} = \{f^{-1}(V) | V \in \mathcal{V}\}$.
- $\mathcal{U}$ is an open cover of X so it has a finite subcover $\{f^{-1}(V_1), \cdots, f^{-1}(V_n)\}$
- So $\{V_1, \cdots, V_n\}$ is a finite subcover of $f(X)$. □

Compactness

- We will show finite products of compact spaces are compact.
- First we need a lemma.

Lemma 242 (The Tube Lemma)

If Y is compact and N is an open subset of $X \times Y$ containing the set $\{x\} \times Y$ for some $x \in X$ then there is an open nbdh W_x of x s.t. $W_x \times Y \subset N$.

Proof.

- $\{x\} \times Y$ is compact since it is the image compact Y under continuous $(c_x \times \mathrm{id}_Y) \circ \Delta_Y : Y \to X \times Y$.
- N is open so it is a union of basis elements of form $U \times V$.
- $\{x\} \times Y$ is a subset of N so these basis elts. cover $\{x\} \times Y$
- Throw out any $U \times V$ with $x \notin U$ and we still get an open cover $\mathcal{U}$ of $\{x\} \times Y$.
- $\mathcal{U}$ has a finite subcover $U_1 \times V_1, \cdots, U_n \times V_n$

Compactness

Proof of Lemma 242 *(continued)*.

- Let $W_x = U_1 \cap \cdots \cap U_n$.
- If $(a, b) \in W_x \times Y$ then there is some i such that $b \in V_i$
- $a \in U_i$ so $(a, b) \in U_i \times V_i \subset N$.
- Thus $W_x \times Y \subset N$

□

Compactness

Proposition 243 (Finite products compact spaces are compact)

If $X_1, \cdots, X_n$ are compact then $X_1 \times \cdots \times X_n$ is compact.

Proof.

- Enough to show product of two compact spaces is compact.
- Suppose X and Y are compact.
- Let $\mathcal{U}$ be an open cover of $X \times Y$.
- $\{x\} \times Y$ is compact so there is a finite subcover $U_1^x, \cdots, U_{n_x}^x$ covering it.
- Let $N_x = U_1^x \cup \cdots \cup U_{n_x}^x$
- By Tube Lemma there is a *tube* $W_x \times Y \subset N_x$.
- $\{W_x\}_{x \in X}$ is an open cover of X so it has a finite subcover $W_{x_1}, \cdots, W_{x_n}$
- Then $W_{x_1} \times Y, \cdots, W_{x_n} \times Y$ covers $X \times Y$
- And each $W_{x_i} \times Y$ is covered by fin. many elts of $\mathcal{U}$. □