Course Info

Reading for Monday, October 22
Chapter 3.27, pgs. 172-177

HW 8 for Monday, October 22

► Chapter 2.24: 3, 5a-d, 8a-d, 12a-f (see pg. 66 for required definitions)
► Chapter 2.25: 1, 2a-c
Components

**Proposition 229**

A space $X$ is locally connected if and only if for every open subset $U \subset X$ each component of $U$ is open in $X$.

**Proof.**

- Suppose $X$ is locally connected.
  - Let $U \subset X$ be open.
  - Let $C \subset U$ be a component of $U$.
  - Let $x \in C$.
  - $x$ has a connected open nbdh $V_x \subset U$.
  - By definition of component $V_x \subset C$.
  - Thus $C = \bigcup_{x \in C} V_x$ is open.
- Suppose for every open subset $U \subset X$ each component of $U$ is open in $X$.
  - Let $U \subset X$ be open.
  - Let $C \subset U$ be the component of $x \in U$.
  - $C$ is a connected open neighborhood of $x$ contained in $U$.

Components

**Proposition 230**

A space $X$ is locally path connected if and only if for every open subset $U \subset X$ each path component of $U$ is open in $X$.

**Proposition 231**

If $X$ is locally path connected then its components are also path components.
Compactness

- Intermediate Value Theorem for $\mathbb{R}$ lead us to notion of connectedness.
- Another classic theorem for continuous functions on $\mathbb{R}$ is the Maximum Theorem:

  \[ \text{If } f : [a, b] \to \mathbb{R} \text{ is continuous then there is } y \in [a, b] \text{ such that } \forall x \in [a, b], f(y) \geq f(x). \]

- What topological property of $[a, b]$ ensures that continuous images achieve a maximum value?
- We want a property $P$ such that if $X$ has property $P$ then for any continuous $f : X \to \mathbb{R}$ there is $y \in X$ such that $\forall x \in X, f(y) \geq f(x)$.
- Note that this property is not shared by $\mathbb{R}$ or $(a, b)$.
- On the other hand it should (probably) be shared by any finite set with the discrete topology.

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**Definition 232 (Cover)**

A cover $\mathcal{A}$ of a topological space $X$ is a collection $\mathcal{A}$ of subsets of $X$ such that $\bigcup \mathcal{A} = X$.

**Definition 233 (Subcover)**

A subcover of a cover $\mathcal{A}$ of a topological space $X$ is a collection $\mathcal{S} \subset \mathcal{A}$ such that $\bigcup \mathcal{S} = X$.

**Definition 234 (Open cover)**

An open cover of a topological space $X$ is a collection $\mathcal{U}$ of open sets such that $\bigcup \mathcal{U} = X$.

**Definition 235 (Compact)**

A topological space $X$ is compact if every open cover of $X$ has a finite subcover.
Compactness

Examples 236 (Noncompact Spaces)
Showing that $X$ is not compact only requires an infinite open cover of $X$ which has no finite subcover:

1. $\mathbb{R}$ is not compact.
   - For each $n \in \mathbb{Z}$ let $U_n = (n, n + \frac{4}{3})$.
   - Let $\mathcal{U} = \{U_n | n \in \mathbb{Z}\}$.
   - $\mathcal{U}$ is an open cover of $\mathbb{R}$ since $\bigcup \mathcal{U} = \mathbb{R}$.
   - Let $\mathcal{S} \subseteq \mathcal{U}$ be a subcover.
   - For all $n \in \mathbb{Z}$, $n + 1 \in U_n$ and $n + 1 \notin U_m$ for $m \neq n$.
   - Hence for each $n \in \mathbb{Z}$ we have $U_n \in \mathcal{S}$.
   - Thus the only subcover of $\mathcal{U}$ is $\mathcal{U}$ which is infinite.

2. $(0, 1)$ is not compact.
   - For each $n \in \mathbb{Z_+}$ let $U_n = (\frac{1}{n}, 1)$.
   - Let $\mathcal{U} = \{U_n | n \in \mathbb{Z_+}\}$.

3. $\mathbb{Q}$ and $\mathbb{Z}$ are not compact.

4. $\mathbb{R}^n$ is not compact.

Compactness

Examples 237 (Compact Spaces)
Showing that $X$ is not compact only requires an infinite open cover of $X$ which has no finite subcover:

1. $\mathbb{R}_f$ is compact.
   - Suppose $\mathcal{U}$ is an open cover of $\mathbb{R}_f$.
   - There is nonempty $U_0 \in \mathcal{U}$.
   - There are finitely many $r_1, \cdots, r_n \in \mathbb{R}_f$ s.t. $r_1, \cdots, r_n \notin U_0$.
   - $\mathcal{U}$ is a cover so there are $U_1, U_2, \cdots, U_n \in \mathcal{U}$ s.t. $r_1 \in U_1, \cdots, r_n \in U_n$.
   - $U_0, U_1, \cdots, U_n$ is a finite subcover of $\mathcal{U}$.

2. $K = \{0\} \cup \{\frac{1}{n} | n \in \mathbb{Z_+}\}$ is compact.
   - Suppose $\mathcal{U}$ is an open cover of $K$.
   - Then there is $U_0 \in \mathcal{U}$ s.t. $0 \in U_0$.
   - There is $N \in \mathbb{Z_+}$ s.t. for $n > N$ $U_0$ contains $\frac{1}{n}$.
   - $\mathcal{U}$ is a cover so there are $U_1, U_2, \cdots, U_n \in \mathcal{U}$ s.t. $\frac{1}{T} \in U_1, \cdots, \frac{1}{N} \in U_N$.
   - $U_0, U_1, \cdots, U_N$ is a finite subcover of $\mathcal{U}$. 

Nathan Broaddus  General Topology and Knot Theory
**Definition 238 (Cover of a subspace)**

If $A$ is a subspace of $X$ a collection of sets covers $A$ if $A \subset \bigcup S$.

**Proposition 239**

A subspace $A \subset X$ is compact if and only if any covering of $A$ by open sets of $X$ has a finite subcovering.

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**Compactness**

What properties do compact sets have?

**Proposition 240 (Compact subsets of Hausdorff spaces are closed)**

If $X$ is Hausdorff and $K \subset X$ is compact then $K$ is closed.

**Proof.**

- Suppose $X$ is Hausdorff and $K \subset X$ is compact.
- Fix $x \in X - K$.
- For each $k \in K$ let $U_k$ be a nbdh of $k$ disjoint from nbdh $V_k$ of $x$.
- $\mathcal{U} = \{ U_k | k \in K \}$ is an open cover of $K$.
- $\mathcal{U}$ has a finite subcover $\{ U_{k_1}, \ldots, U_{k_n} \}$.
- So $V_x = \bigcap_{i=1}^n V_{k_i}$ is an open nbhd of $x$ disjoint from $K$.
- So $X - K = \bigcup_{x \in X - K} V_x$ is open.

□
Compactness

Tools for proving compactness

Proposition 241 (Continuous images of compact spaces are compact)

If $X$ and $Y$ are spaces and $X$ is compact and $f : X \to Y$ is continuous then $f(X)$ is a compact subspace of $Y$.

Proof.

Suppose $X$ is compact and $f : X \to Y$ is continuous

Let $\mathcal{V}$ be a cover of $f(X)$ by open sets in $Y$.

Let $\mathcal{U} = \{f^{-1}(V) | V \in \mathcal{V}\}$.

$\mathcal{U}$ is an open cover of $X$ so it has a finite subcover $\{f^{-1}(V_1), \ldots, f^{-1}(V_n)\}$

So $\{V_1, \ldots, V_n\}$ is a finite subcover of $f(X)$.

We will show finite products of compact spaces are compact.

First we need a lemma.

Lemma 242 (The Tube Lemma)

If $Y$ is compact and $N$ is an open subset of $X \times Y$ containing the set $\{x\} \times Y$ for some $x \in X$ then there is an open nbhd $W_x$ of $x$ s.t. $W_x \times Y \subset N$.

Proof.

$\{x\} \times Y$ is compact since it is the image compact $Y$ under continuous $(c_x \times \text{id}_Y) \circ \Delta_Y : Y \to X \times Y$.

$N$ is open so it is a union of basis elements of form $U \times V$.

$\{x\} \times Y$ is a subset of $N$ so these basis elts. cover $\{x\} \times Y$.

Throw out any $U \times V$ with $x \notin U$ and we still get an open cover $\mathcal{U}$ of $\{x\} \times Y$.

$\mathcal{U}$ has a finite subcover $U_1 \times V_1, \ldots, U_n \times V_n$.
Compactness

Proof of Lemma 242 (continued).

- Let $W_x = U_1 \cap \cdots \cap U_n$.
- If $(a, b) \in W_x \times Y$ then there is some $i$ such that $b \in V_i$.
- $a \in U_i$ so $(a, b) \in U_i \times V_i \subset N$.
- Thus $W_x \times Y \subset N$.

Proposition 243 (Finite products compact spaces are compact)

If $X_1, \cdots, X_n$ are compact then $X_1 \times \cdots \times X_n$ is compact.

Proof.

- Enough to show product of two compact spaces is compact.
- Suppose $X$ and $Y$ are compact.
- Let $\mathcal{U}$ be an open cover of $X \times Y$.
- $\{x\} \times Y$ is compact so there is a finite subcover $U_1^x, \cdots, U_n^x$ covering it.
- Let $N_x = U_1^x \cup \cdots \cup U_n^x$.
- By Tube Lemma there is a tube $W_x \times Y \subset N_x$.
- $\{W_x\}_{x \in X}$ is an open cover of $X$ so it has a finite subcover $W_{x_1}, \cdots, W_{x_n}$.
- Then $W_{x_1} \times Y, \cdots, W_{x_n} \times Y$ covers $X \times Y$.
- And each $W_{x_i} \times Y$ is covered by fin. many elts of $\mathcal{U}$.