Math 6802
Algebraic Topology II

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Eilenberg-Steenrod Axioms

Axiomatic approach:

1. List axioms satisfied by homology theory
2. Assume such a theory exists
3. Compute homology groups using axioms
4. Define singular homology theory
5. Show it satisfies the axioms *(very technical)*

Definition 1 (Homology theory)

A **homology theory** (h_*, ∂) is a series of functors $h_n : \textbf{TopPair} \to \textbf{Ab}$ and for each pair (X, A) natural transformations

$$\partial_n(X, A) : h_n(X, A) \to h_{n-1}(A)$$

called "boundary maps" satisfying

A1. Homotopy Axiom

A2. Exactness Axiom

A3. Excision Axiom

A4. Additivity Axiom

We will also assume

A5. Dimension Axiom

Axiom A1. Homotopy

If $f, g : (X, A) \to (Y, B)$ are homotopic maps

then homorphisms

$$h_n(f) : h_n(X, A) \to h_n(Y, B)$$

and

$$h_n(g) : h_n(X, A) \to h_n(Y, B)$$

are equal.

In other words modifying $f : (X, A) \to (Y, B)$ by a homotopy preserves $h_n(f)$.

Proof Idea.

For singular simplicial homology use homotopy from f to g to construct **chain homotopy** from ∂_f to ∂_g □

Axiom A2. Exactness

For any pair (X, A) let

$$i : A \to X$$
$$j : X \to (X, A)$$

be the inclusion maps.

The sequence

$$\cdots \xrightarrow{\partial_{n+1}(X,A)} h_n(A) \xrightarrow{h_n(i)} h_n(X) \xrightarrow{h_n(j)} h_n(X, A) \xrightarrow{\partial_n(X,A)} h_{n-1}(A) \xrightarrow{h_{n-1}(i)} h_{n-1}(X) \xrightarrow{h_{n-1}(j)} \cdots$$

is exact.

Proof Idea.

This is content of Snake Lemma. □

Axiom A3. Excision

For any pair (X, A) if $U \subset A$ is open with $\overline{U} \subset \operatorname{int} A$ then the inclusion

$$i : (X - U, A - U) \to (X, A)$$

induces an isomorphism

$$h_n(i) : h_n(X - U, A - U) \cong h_n(X, A)$$

Allows us to "excise" U from homology calculations

Proof Idea.

Show there is chain homotopy to chain complex where simplices have been barycentricaly subdivided. □

Axiom A4. Additivity

X_α disjoint spaces. Then

$$h_n\left(\coprod X_\alpha\right) = \bigoplus_\alpha h_n(X_\alpha)$$

Proof Idea.

Immediate from additivity of homology for chain complexes. □

Axiom A5. Dimension *(not required for a homology theory)*

For the one point space $\{*\}$

$$h_n(\{*\}) = \begin{cases} \mathbf{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Later we will relax Axiom A5 slightly to allow:

Axiom A5′

For the one point space $\{*\}$

$$h_n(\{*\}) = \begin{cases} A, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

For some abelian group A.

Proof Idea.

This is one of few spaces whose homology groups are computable from definition. □

Theorem 2

Singular simplicial homology theory (H_*, ∂) *satisfies 1-5.*

Stable Homotopy Theory

Supension gives

$$\pi_n(X) \xrightarrow{S} \pi_{n+1}(SX) \xrightarrow{S} \pi_{n+2}(S^2X) \xrightarrow{S} \cdots$$

Definition 3 (Stable homotopy group)

The stable homotopy group is

$$\pi_n^S(X) = \varinjlim_k \pi_{n+k}(S^k X)$$

Theorem 4

(Relative) stable homotopy groups (π_*^S, ∂^S) *satisfy 1-4.*