Math 6802
Algebraic Topology II

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Naturality of the connecting homomorphism

- Suppose we have a commutative diagram of chain complexes and chain maps with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma} & \circlearrowleft & \downarrow{\scriptstyle \delta} & \circlearrowleft & \downarrow{\scriptstyle \varepsilon} & & \\
0 & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' & \longrightarrow & 0
\end{array}$$

- The Snake Lemma gives

$$\begin{array}{ccccccccccc}
\cdots \to & H_{n+1}(E) & \to & H_n(C) & \to & H_n(D) & \to & H_n(E) & \to & H_{n-1}(C) & \to \cdots \\
\circlearrowleft & \downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle \gamma_*} & \circlearrowleft & \downarrow{\scriptstyle \delta_*} & \circlearrowleft & \downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle \gamma_*} & \circlearrowleft \\
\cdots \to & H_{n+1}(E') & \to & H_n(C') & \to & H_n(D') & \to & H_n(E') & \to & H_{n-1}(C') & \to \cdots
\end{array}$$

- Fact that $H_n : \mathbf{Chain} \to \mathbf{Ab}$ is a functor shows some squares commute.

Theorem 1 (Naturality of connecting homomorphism)

Given a commutative diagram of chain complexes and chain maps with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow \gamma & \circlearrowleft & \downarrow \delta & \circlearrowleft & \downarrow \varepsilon & & \\ 0 & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' & \longrightarrow & 0 \end{array}$$

The following diagram commutes:

$$\begin{array}{ccc} H_{n+1}(E) & \xrightarrow{\partial} & H_n(C) \\ \downarrow \varepsilon_* & \circlearrowleft & \downarrow \gamma_* \\ H_{n+1}(E') & \xrightarrow{\partial} & H_n(C') \end{array}$$

Note on category theory:

Definition 2 (Natural transformation)

- Given two functors $F, G : \mathcal{C} \to \mathcal{D}$
- A **natural transformation** η from F to G assigns a morphism $\eta_C : F(C) \to G(C)$ for each $C \in \mathrm{Ob}(\mathcal{C})$ such that the following diagram commute for all morphisms $f \in \mathrm{Mor}_{\mathcal{C}}(C_1, C_2)$

$$\begin{array}{ccc} F(C_1) & \xrightarrow{F(f)} & F(C_2) \\ \big\downarrow{\scriptstyle \eta_{C_1}} & \circlearrowleft & \big\downarrow{\scriptstyle \eta_{C_2}} \\ G(C_1) & \xrightarrow{G(f)} & G(C_2) \end{array}$$

How does our naturality fit into this picture?

- Let $\mathcal{C} = \textbf{SESChain}$ be the category of short exact sequences of chain complexes $C \hookrightarrow D \twoheadrightarrow E$ with morphisms given by commuting diagrams of chain maps from one short exact sequence to another.
- Let $\mathcal{D} = \textbf{Ab}$
- Let $F(C \hookrightarrow D \twoheadrightarrow E) = H_n(E)$
- Let $G(C \hookrightarrow D \twoheadrightarrow E) = H_{n-1}(C)$
- Let $\eta_{(C \hookrightarrow D \twoheadrightarrow E)} = \partial : H_n(E) \to H_{n-1}(C)$

Or

- Let $\mathcal{C} = \textbf{TopPair}$
- Let $\mathcal{D} = \textbf{Ab}$
- Let $F(X, A) = H_n(X, A)$
- Let $G(X, A) = H_{n-1}(A)$
- Let $\eta_{(X,A)} = \partial : H_n(X, A) \to H_{n-1}(A)$

Review of tensor products

Definition 3 (Bilinear function)

A, B, C abelian groups

$$\varphi : A \times B \to C$$

is **bilinear** if

$$\varphi(a_1 + a_2, b) = \varphi(a_1, b) + \varphi(a_2, b)$$

and

$$\varphi(a, b_1 + b_2) = \varphi(a, b_1) + \varphi(a, b_2)$$

Definition 4 (Tensor product of abelian groups)

A, B abelian groups. The **tensor product** of A and B is the group

$$A \otimes B = \frac{\mathbf{Z}[A \times B]}{\left\langle \begin{array}{c} (a, b_1 + b_2) - (a, b_1) - (a, b_2) \\ (a_1 + a_2, b) - (a_1, b) - (a_2, b) \end{array} \right\rangle}$$

The class of (a, b) in $A \otimes B$ is written $a \otimes b$.

Note:

- $\{(a, b)\}$ generates $\mathbf{Z}[A \times B]$
- so $\{a \otimes b\}$ generates $A \otimes B$
- but just as general element of $\mathbf{Z}[A \times B]$ is of the form

$$\sum_{i=1}^{k} n_i (a_i, b_i)$$

- general element of $A \otimes B$ is of the form

$$\sum_{i=1}^{k} n_i \cdot a_i \otimes b_i$$

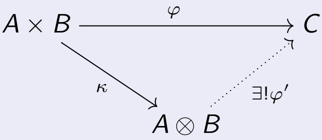
Lemma 5 (Universal property of the tensor product)

A, B abelian groups. Let

$$\kappa : A \times B \to A \otimes B$$

be the map $\kappa(a, b) = a \otimes b$.

- *Then κ is bilinear*
- *for any* bilinear $\varphi : A \times B \to C$
- *there exists a unique* homomorphism $\varphi' : A \otimes B \to C$
- *such that $\varphi = \varphi' \circ \kappa$.*



Lemma 6 (Tensor product of homomorphisms)

Given homomorphisms of abelian groups

$$f : A \to A'$$

$$g : B \to B'$$

There is unique homomorphism

$$f \otimes g : A \otimes B \to A' \otimes B'$$

satisfying

$$(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$$

Idea of proof.

Apply universal property to the map $\varphi : A \times B \to A' \otimes B'$ given by $\varphi(a, b) = f(a) \otimes g(b)$ $\square$