Math 6802
Algebraic Topology II

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Proposition 1 (Properties of the tensor product)

1. $$(f_2 \circ f_1) \otimes (g_2 \circ g_1) = (f_2 \otimes g_2) \circ (f_1 \otimes g_1)$$

   That is, the following diagram commutes

   $$A_1 \otimes B_1 \xrightarrow{f_1 \otimes g_1} A_2 \otimes B_2 \xrightarrow{f_2 \otimes g_2} A_3 \otimes B_3$$

   $$A_1 \otimes B_1 \xrightarrow{(f_2 \circ f_1) \otimes (g_2 \circ g_1)} A_3 \otimes B_3$$

2. $$(\oplus_\alpha A_\alpha) \otimes B \cong \oplus_\alpha (A_\alpha \otimes B)$$

3. $$A \otimes B \cong B \otimes A \;\; \textit{via the map } a \otimes b \mapsto b \otimes a$$

4. $$\mathbf{Z} \otimes A \cong A \;\; \textit{via the map } 1 \otimes a \mapsto a$$

5. *For sets S and T*

   $$\mathbf{Z}[S] \otimes \mathbf{Z}[T] \cong \mathbf{Z}[S \times T] \;\; \textit{via the map } s \otimes t \mapsto (s, t)$$

Proposition 2

If $A' \subset A$ and $B' \subset B$ are abelian groups then

$$(A/A') \otimes (B/B') \cong \frac{A \otimes B}{i_{A'} \otimes \mathbf{1}_B(A' \otimes B) + \mathbf{1}_A \otimes i_{B'}(A \otimes B')}$$

Proof.

- Let
$$q_1 : A \to A/A'$$
$$q_2 : B \to B/B'$$
be the quotient maps
- Then we have
$$q_1 \otimes q_2 : A \otimes B \to (A/A') \otimes (B/B')$$
- Claim 1: $A' \otimes B \subset \ker q_1 \otimes q_2$. If $a' \in A'$ and $b \in B$ then
$$\begin{aligned} q_1 \otimes q_2(a' \otimes b) &= q_1(a') \otimes q_2(b) \\ &= 0 \otimes q_2(b) = 0 \end{aligned}$$

Proof of Proposition 2 *(continued)*.

$$q_1 : A \to A/A'$$

$$q_2 : B \to B/B'$$

- Similarly $A \otimes B' \subset \ker q_1 \otimes q_2$.
- Let $K = i_{A'} \otimes \mathbf{1}_B(A' \otimes B) + \mathbf{1}_A \otimes i_{B'}(A \otimes B')$
- $q_1 \otimes q_2(K) = 0$ so $q_1 \otimes q_2$ induces homomorphism

$$Q : (A \otimes B)/K \to (A/A') \otimes (B/B')$$

- Define

$$r : (A/A') \times (B/B') \to (A \otimes B)/K$$

by setting

$$r([a], [b]) = [a \otimes b]$$

- r is well-defined and bilinear.

Proof of Proposition 2 *(continued)*.

$$q_1 : A \to A/A'$$
$$q_2 : B \to B/B'$$

- By universal property get homomorphism

$$R : (A/A') \otimes (B/B') \to (A \otimes B)/K$$

with

$$R([a] \otimes [b]) = [a \otimes b]$$

- $QR = \mathbf{1}$ and $RQ = \mathbf{1}$
- Hence

$$(A/A') \otimes (B/B') \cong \frac{A \otimes B}{i_{A'} \otimes \mathbf{1}_B(A' \otimes B) + \mathbf{1}_A \otimes i_{B'}(A \otimes B')}$$

□

Example 3

$$
\begin{aligned}
(\mathbf{Z}/n\mathbf{Z}) \otimes (\mathbf{Z}/m\mathbf{Z}) &\cong \frac{\mathbf{Z} \otimes \mathbf{Z}}{n\mathbf{Z} \otimes \mathbf{Z} + \mathbf{Z} \otimes m\mathbf{Z}} \\
&\cong \frac{\mathbf{Z}}{n\mathbf{Z} + m\mathbf{Z}} \\
&\cong \mathbf{Z}/\gcd(n, m)\mathbf{Z}
\end{aligned}
$$

For example

$$(\mathbf{Z}/6\mathbf{Z}) \otimes (\mathbf{Z}/6\mathbf{Z}) \cong \mathbf{Z}/6\mathbf{Z}$$

$$(\mathbf{Z}/15\mathbf{Z}) \otimes (\mathbf{Z}/12\mathbf{Z}) \cong \mathbf{Z}/3\mathbf{Z}$$

$$(\mathbf{Z}/9\mathbf{Z}) \otimes (\mathbf{Z}/8\mathbf{Z}) \cong \mathbf{Z}/\mathbf{Z} = 0$$

Right exactness of tensor product

Proposition 4 (Right exactness of tensor product)

If

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is exact then for all D

$$A \otimes D \xrightarrow{\alpha \otimes \mathbf{1}_D} B \otimes D \xrightarrow{\beta \otimes \mathbf{1}_D} C \otimes D \to 0$$

is exact.

Proof of Proposition 4.

- Define
$$\varphi : C \times D \to \frac{B \otimes D}{\alpha \otimes \mathbf{1}_D(A \otimes D)}$$
by setting $\varphi(\beta(b), d) = [b \otimes d]$
- Claim: φ is well-defined

Proof of Proposition 4 *(continued)*.

- Suppose $\beta(b) = \beta(b')$
- Then $b - b' \in \ker \beta$ thus there is $a \in A$ with $\alpha(a) = b - b'$
- Thus

$$\begin{aligned} b \otimes d - b' \otimes d &= (b - b') \otimes d \\ &= \alpha(a) \otimes d \\ &= \alpha \otimes \mathbf{1}_D(a \otimes d) \end{aligned}$$

- so φ is well-defined.
- Claim 2: φ is bilinear

$$\begin{aligned} \varphi(\beta(b) + \beta(b'), d) &= \varphi(\beta(b + b'), d) \\ &= [(b + b') \otimes d] \\ &= [b \otimes d + b' \otimes d] \\ &= [b \otimes d] + [b' \otimes d] \\ &= \varphi(\beta(b), d) + \varphi(\beta(b'), d) \end{aligned}$$

Proof of Proposition 4 *(continued)*.

- $$\begin{aligned}\varphi(\beta(b), d + d') &= [b \otimes (d + d')] \\ &= [b \otimes d + b \otimes d'] \\ &= \varphi(\beta(b), d) + \varphi(\beta(b), d')\end{aligned}$$

- Universal property of tensor product gives

$$\Phi : C \otimes D \to \frac{B \otimes D}{\alpha \otimes \mathbf{1}_D(A \otimes D)}$$

- Claim 3: $\beta \otimes \mathbf{1}_D(\alpha \otimes \mathbf{1}_D(A \otimes D)) = 0$

$$\begin{aligned}\beta \otimes \mathbf{1}_D(\alpha \otimes \mathbf{1}_D(a \otimes d)) &= (\beta \circ \alpha) \otimes (\mathbf{1}_D \circ \mathbf{1}_D)(a \otimes d) \\ &= \beta\alpha(a) \otimes d \\ &= 0 \otimes d = 0\end{aligned}$$

Proof of Proposition 4 *(continued)*.

- So
$$\beta \otimes \mathbf{1}_D : B \otimes D \to C \otimes D$$
induces
$$\Theta : \frac{B \otimes D}{\alpha \otimes \mathbf{1}_D(A \otimes D)} \to C \otimes D$$
- $\Theta\Phi(\beta(b) \otimes d) = \Theta([b \otimes d]) = \beta(b) \otimes d$
- $\Phi\Theta([b \otimes d]) = \Phi(\beta(b) \otimes d) = [b \otimes d]$
- So Φ is an isomorphism establishing exactness at $B \otimes D$.
- Surjectivity of β gives surjectivity of $\beta \otimes \mathbf{1}_D$ establishing exactness at $C \otimes D$.

□

Example 5 (Tensor product is not left exact)

- Consider the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\times 5} \mathbf{Z} \to \mathbf{Z}/5\mathbf{Z}$$

- Tensor with $\mathbf{Z}/5\mathbf{Z}$ to get

$$0 \to \mathbf{Z} \otimes (\mathbf{Z}/5\mathbf{Z}) \xrightarrow{(\times 5)\otimes \mathbf{1}} \mathbf{Z} \otimes (\mathbf{Z}/5\mathbf{Z}) \to \mathbf{Z}/5\mathbf{Z} \otimes \mathbf{Z}/5\mathbf{Z}$$

- which becomes

$$0 \to \mathbf{Z}/5\mathbf{Z} \xrightarrow{0} \mathbf{Z}/5\mathbf{Z} \xrightarrow{\cong} \mathbf{Z}/5\mathbf{Z}$$

- No longer exact at leftmost $\mathbf{Z}/5\mathbf{Z}$

Tensor products and chain complexes

Given a chain complex (C, ∂)

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \cdots$$

and an abelian group G we get a new chain complex $(C \otimes G, \partial \otimes \mathbf{1}_G)$.

$$\cdots \xrightarrow{\partial_{n+1}\otimes\mathbf{1}_G} C_n \otimes G \xrightarrow{\partial_n\otimes\mathbf{1}_G} C_{n-1} \otimes G \xrightarrow{\partial_{n-1}\otimes\mathbf{1}_G} C_{n-2} \otimes G \xrightarrow{\partial_{n-2}\otimes\mathbf{1}_G} \cdots$$

Lemma 6

If C is a chain complex then $C \otimes G$ is a chain complex.

Proof.

$$(\partial_n \otimes \mathbf{1}_G) \circ (\partial_{n-1} \otimes \mathbf{1}_G) = (\partial_n \circ \partial_{n-1}) \otimes \mathbf{1}_G = 0 \otimes \mathbf{1}_G = 0$$

□

Lemma 7

If

$$f : C \to D$$

is a chain map then

$$f \otimes \mathbf{1}_G : C \otimes G \to D \otimes G$$

is a chain map.

Proof.

$f : C \to D$ is a chain map so $\partial f = f \partial$

$$\begin{aligned}(f \otimes \mathbf{1}_G) \circ (\partial \otimes \mathbf{1}_G) &= (f \circ \partial) \otimes \mathbf{1}_G \\ &= (\partial \circ f) \otimes \mathbf{1}_G \\ &= (\partial \otimes \mathbf{1}_G) \circ (f \otimes \mathbf{1}_G)\end{aligned}$$

□

Lemma 8

If

$$T : C \to D$$

is a chain homotopy between chain maps $f, g : C \to D$ *then*

$$T \otimes \mathbf{1}_G : C \otimes G \to D \otimes G$$

is a chain homotopy between chain maps $f \otimes \mathbf{1}_G$ *and* $g \otimes \mathbf{1}_G$

Proof.

$T : C \to D$ is a chain homotopy so $\partial T + T\partial = g - f$

$$\begin{aligned}(\partial \otimes \mathbf{1}) \circ (T \otimes \mathbf{1}) + (T \otimes \mathbf{1}) \circ (\partial \otimes \mathbf{1}) &= (\partial T) \otimes \mathbf{1} + (T\partial) \otimes \mathbf{1} \\ &= (\partial T + T\partial) \otimes \mathbf{1} \\ &= (g - f) \otimes \mathbf{1} \\ &= g \otimes \mathbf{1} - f \otimes \mathbf{1}\end{aligned}$$

□

Homology with coefficients in G

Definition 9 (Homology with coefficients in G)

Given a chain complex C let

$$H_n(C;G) = H_n(C \otimes G)$$

If X is a space let

$$H_n(X;G) = H_n(C(X) \otimes G)$$

$$\tilde{H}_n(X;G) = H_n(\tilde{C}(X) \otimes G)$$

$$H_n^{\mathrm{cw}}(X;G) = H_n(C^{\mathrm{cw}}(X) \otimes G)$$

For (X, A) a topological pair let

$$H_n(X, A; G) = H_n(C(X, A) \otimes G)$$

Example 10 (Homology of $\mathbf{R}\mathrm{P}^4$)

Standard CW chain complex for $\mathbf{R}\mathrm{P}^4$ is

$$\cdots \to 0 \to \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{\times 0} \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{\times 0} \mathbf{Z} \xrightarrow{\times 0} 0$$

Tensoring with $\mathbf{Z}_2$ we get $C^{\mathrm{cw}}(\mathbf{R}\mathrm{P}^4) \otimes \mathbf{Z}_2$

$$\cdots \to 0 \to \mathbf{Z}_2 \xrightarrow{\times 0} \mathbf{Z}_2 \xrightarrow{\times 0} \mathbf{Z}_2 \xrightarrow{\times 0} \mathbf{Z}_2 \xrightarrow{\times 0} \mathbf{Z}_2 \xrightarrow{\times 0} 0$$

So

$$H_n^{\mathrm{cw}}(\mathbf{R}\mathrm{P}^4; \mathbf{Z}_2) \cong \begin{cases} \mathbf{Z}_2, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Tensoring with $\mathbf{Q}$ we get $C^{\mathrm{cw}}(\mathbf{R}\mathrm{P}^4) \otimes \mathbf{Q}$

$$\cdots \to 0 \to \mathbf{Q} \xrightarrow[\cong]{\times 2} \mathbf{Q} \xrightarrow{\times 0} \mathbf{Q} \xrightarrow[\cong]{\times 2} \mathbf{Q} \xrightarrow{\times 0} \mathbf{Q} \xrightarrow{\times 0} 0$$

So

$$H_n^{\mathrm{cw}}(\mathbf{R}\mathrm{P}^4; \mathbf{Q}) \cong \begin{cases} \mathbf{Q}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Note that $C \otimes \mathbf{Z} = C$ so

$$H_n(C; \mathbf{Z}) \cong H_n(C)$$

Why consider homology with other coefficent groups?

1. If k is a field then $H_n(C; k)$ is a vector space over k.
2. Computation of $H_n(C; k)$ are often easier than $H_n(C)$.

What advantages does the coefficient group $\mathbf{Z}$ have?

1. Homology groups with coefficients in $\mathbf{Z}$ are the strongest topological invariants

Homology with coefficients in **Z** are "universal" since they determine homology with coefficients in G for all abelian groups G.

Theorem 11 (Universal coefficient theorem)

For C a chain complex there is a split short exact sequence

$$0 \to H_n(C) \otimes G \to H_n(C; G) \to \operatorname{Tor}(H_{n-1}(C), G) \to 0.$$

This sequence is natural in that if $C \to D$ is a chain map then we get commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_n(C) \otimes G & \longrightarrow & H_n(C; G) & \longrightarrow & \operatorname{Tor}(H_{n-1}(C), G) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_n(D) \otimes G & \longrightarrow & H_n(D; G) & \longrightarrow & \operatorname{Tor}(H_{n-1}(D), G) & \longrightarrow & 0 \end{array}$$

The Tor functor

Definition 12 (Free resolution)

A **free resolution** of the abelian group A is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to A \to 0$$

where each F_i is free abelian.

Example 13 (Two free resolutions of $\mathbf{Z}_5$)

$$\cdots \to 0 \to 0 \to \mathbf{Z} \xrightarrow{\times 5} \mathbf{Z} \to \mathbf{Z}_5 \to 0$$

$$\cdots \xrightarrow{\times 1} \mathbf{Z} \xrightarrow{\times 0} \mathbf{Z} \xrightarrow{\times 1} \mathbf{Z} \xrightarrow{\times 0} \mathbf{Z} \xrightarrow{\times 5} \mathbf{Z} \to \mathbf{Z}_5 \to 0$$

Theorem 14 (Existence of free resolutions)

Every abelian group A has a free resolution

$$\cdots \to 0 \to R \to F \to A \to 0$$

where F and R are free abelian.

Theorem 15

Every subgroup of a free abelian group is a free abelian group.

Proof of Theorem 14.

We have a surjection

$$\mathbf{Z}[A] \xrightarrow{p} A$$

given by $p(a) = a$. Let $F = \mathbf{Z}[A]$ and $R = \ker p$.

Then

$$\cdots \to 0 \to R \to F \xrightarrow{p} A \to 0$$

is a free resolution of A. □