Math 6802
Algebraic Topology II

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The Tor functor

Definition 1 (Free resolution)

A **free resolution** of the abelian group A is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to A \to 0$$

where each F_i is free abelian.

Example 2 (Two free resolutions of $\mathbf{Z}_5$)

$$\cdots \to 0 \to 0 \to \mathbf{Z} \xrightarrow{\times 5} \mathbf{Z} \to \mathbf{Z}_5 \to 0$$

$$\cdots \xrightarrow{\times 1} \mathbf{Z} \xrightarrow{\times 0} \mathbf{Z} \xrightarrow{\times 1} \mathbf{Z} \xrightarrow{\times 0} \mathbf{Z} \xrightarrow{\times 5} \mathbf{Z} \to \mathbf{Z}_5 \to 0$$

Theorem 3 (Existence of free resolutions)

Every abelian group A has a free resolution

$$\cdots \to 0 \to R \to F \to A \to 0$$

where F and R are free abelian.

Theorem 4

Every subgroup of a free abelian group is a free abelian group.

Proof of Theorem 3.

We have a surjection

$$\mathbf{Z}[A] \xrightarrow{p} A$$

given by $p(a) = a$. Let $F = \mathbf{Z}[A]$ and $R = \ker p$.

Then

$$\cdots \to 0 \to R \to F \xrightarrow{p} A \to 0$$

is a free resolution of A. □

Definition 5 (The Tor functor)

Let A and B be abelian groups and

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to A \to 0$$

be a free resolution of A.

Let $\partial_0 : F_0 \to 0$ be the 0 map so we have a chain complex (F, ∂)

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0$$

Tensoring with B we get the chain complex $F \otimes B$

$$\cdots \xrightarrow{\partial_3 \otimes \mathbf{1}_B} F_2 \otimes B \xrightarrow{\partial_2 \otimes \mathbf{1}_B} F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B \xrightarrow{\partial_0 \otimes \mathbf{1}_B} 0$$

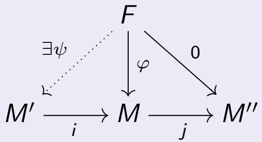
$$\mathrm{Tor}_n(A, B) = H_n(F \otimes B)$$

Theorem 6 (Tor is well-defined)

$\mathrm{Tor}_n(A, B)$ *is independent of the free resolution* F *of* A.

Lemma 7 (Free abelian groups are projective)

Suppose we have commutative



1. *If F is a free abelian group*
2. $M' \xrightarrow{i} M \xrightarrow{j} M''$ *is exact*

There is a homomorphism ψ making the diagram commute.

Proof.

- Let $\{e_\alpha\}$ be a free basis for F
- $j\varphi(e_\alpha) = 0(e_\alpha) = 0$ so $\varphi(e_\alpha) \in \ker j = \operatorname{Im} i$
- Thus there is $m'_\alpha \in M'$ such that $i(m'_\alpha) = \varphi(e_\alpha)$
- Let $\psi(e_\alpha) = m'_\alpha$

□

We will prove Theorem 6 using the following lemma

Lemma 8

If F and F' are two free resolutions of A then there is a chain map $f : F \to F'$ which is a chain homotopy equivalence.

Proof.

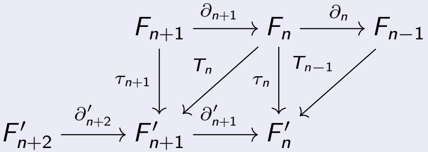
- We have

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial_2} & F_1 & \xrightarrow{\partial_1} & F_0 & \xrightarrow{\varepsilon} & A & \xrightarrow{0} & 0 \\ & & & & & & \downarrow{\scriptstyle \mathbf{1}_A} & & \\ \cdots & \xrightarrow{\partial_2'} & F_1' & \xrightarrow{\partial_1'} & F_0' & \xrightarrow{\varepsilon'} & A & \xrightarrow{0} & 0 \end{array}$$

- Apply Lemma 7 (free ab. grps. are proj.) with $\varphi = \mathbf{1}_A \circ \varepsilon$
- Get $\psi : F_0 \to F_0'$ with $\varepsilon' \psi = \mathbf{1}_A \varepsilon$
- Let $f_0 = \psi$
- Now apply Lemma 7 with $\varphi = f_0 \partial_1$ to get $f_1 : F_1 \to F_1'$
- Continue inductively to get chain map $f : F \to F'$

Proof of Lemma 8 *(continued)*.

- Now we will show that the chain homotopy type of f is unique.
- That is if $g : F \to F'$ is another chain map extending $\mathbf{1}_A : A \to A$ then there is a chain homotopy T between f and g
- Let $\tau = g - f$
- Assume inductively that $\partial'_{n+1} T_n + T_{n-1}\partial_n = \tau_n$
- We have (non-commutating diagram)



- Consider the map $(\tau_{n+1} - T_n\partial_{n+1}) : F_{n+1} \to F'_{n+1}$

$$\begin{aligned}
\partial'_{n+1}(\tau_{n+1} - T_n\partial_{n+1}) &= \partial'_{n+1}\tau_{n+1} - \partial'_{n+1}T_n\partial_{n+1} \\
&= \partial'_{n+1}\tau_{n+1} - (\tau_n - T_{n-1}\partial_n)\partial_{n+1} \\
&= \partial'_{n+1}\tau_{n+1} - \tau_n\partial_{n+1} + T_{n-1}\partial_n\partial_{n+1} \\
&= 0
\end{aligned}$$

Proof of Lemma 8 *(continued)*.

- Apply Lemma 7 (free ab. grps. are proj.) with $\varphi = \tau_{n+1} - T_n\partial_{n+1}$
- Let $T_{n+1} = \psi : F_{n+1} \to F'_{n+2}$
- Then $\partial'_{n+2} T_{n+1} = \varphi = \tau_{n+1} - T_n\partial_{n+1}$
- Hence
$$\partial'_{n+2} T_{n+1} + T_n\partial_{n+1} = \tau_{n+1} = g_{n+1} - f_{n+1}$$
- We may start the induction in degree -2 where all groups are 0.
- Thus the chain homotopy class of $f : F \to F'$ is unique.
- Similarly we get $f' : F' \to F$
- $\mathbf{1}_F : F \to F$ extends $\mathbf{1}_A : A \to A$
- so $f' \circ f$ is chain homotopic to $\mathbf{1}_F$
- That is, f is a chain homotopy equivalence.

□

Now we finish proof that $\text{Tor}_n(A, B)$ is independent of free resolution of A.

Proof of Theorem 6 (Tor is well-defined).

- Let F and F' be two free resolutions of A
- By Lemma 8 we have a chain homotopy equivalence

$$f : F \to F'$$

- Hence there is a chain map $f' : F' \to F$ such that $f' \circ f$ is chain homotopic to $\mathbf{1}_F$
- Tensoring with B we get $(f' \otimes \mathbf{1}_B) \circ (f \otimes \mathbf{1}_B)$ is chain homotopic to $\mathbf{1}_F \otimes B$
- Thus

$$(f \otimes \mathbf{1}_B)_* : H_n(F \otimes B) \to H_n(F' \otimes B)$$

  is an isomorphism
- $\text{Tor}_n(A, B) = H_n(F \otimes B) \cong H_n(F' \otimes B)$ is well-defined

□

- As we saw in Lemma 3 every abelian group A has a free resolution

$$\cdots \to 0 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \to 0$$

- So we have the chain complexs F

$$\cdots \to 0 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0$$

- and $F \otimes B$

$$\cdots \to 0 \xrightarrow{\partial_2 \otimes \mathbf{1}_B} F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B \xrightarrow{\partial_0 \otimes \mathbf{1}_B} 0$$

- $$\mathrm{Tor}_n(A, B) = H_n(F \otimes B) = \begin{cases} \frac{F_0 \otimes B}{\mathrm{Im}(\partial_1 \otimes \mathbf{1}_B)}, & n = 0 \\ \ker(\partial_1 \otimes \mathbf{1}_B), & n = 1 \\ 0, & n \neq 0, 1 \end{cases}$$

- We can say more about $\mathrm{Tor}_0(A, B)$.

- We have the exact sequence

$$F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \to 0$$

- Which remains exact after tensoring with B so we get exact

$$F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes \mathbf{1}_B} A \otimes B \to 0$$

- Hence

$$\mathrm{Tor}_0(A, B) \cong \frac{F_0 \otimes B}{\mathrm{Im}(\partial_1 \otimes \mathbf{1}_B)} \cong A \otimes B$$

- Since $\mathrm{Tor}_1(A, B)$ is the only (possibly) new object we define

Definition 9

$$\mathrm{Tor}(A, B) = \mathrm{Tor}_1(A, B)$$

- Note that if we have exact

$$0 \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \to 0$$

- Then we have exact

$$F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes \mathbf{1}_B} A \otimes B \to 0$$

- and hence exact

$$0 \to \ker(\partial_1 \otimes \mathbf{1}_B) \to F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes \mathbf{1}_B} A \otimes B \to 0$$

- and hence exact

$$0 \to \mathrm{Tor}(A, B) \to F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes \mathbf{1}_B} A \otimes B \to 0$$

- In particular if $\mathrm{Tor}(A, B) = 0$ then tensoring with B preserves exactness.

Example 10

Let's compute $\mathrm{Tor}(\mathbf{Z}_{60}, \mathbf{Z}_{42})$

- Free resolution F of $\mathbf{Z}_{60}$

$$\cdots \to 0 \to \mathbf{Z} \xrightarrow{\times 60} \mathbf{Z} \to \mathbf{Z}_{60} \to 0$$

- $F \otimes \mathbf{Z}_{42}$ is

$$0 \to \mathbf{Z} \otimes \mathbf{Z}_{42} \xrightarrow{(\times 60) \otimes \mathbf{1}} \mathbf{Z} \otimes \mathbf{Z}_{42} \to 0$$

- Simplifying

$$0 \to \mathbf{Z}_{42} \xrightarrow{\times 60} \mathbf{Z}_{42} \to 0$$

- Hence

$$\mathrm{Tor}(\mathbf{Z}_{60}, \mathbf{Z}_{42}) \cong \ker(\times 60) \cong \frac{7\mathbf{Z}}{42\mathbf{Z}} \cong \mathbf{Z}_6 \cong \frac{\mathbf{Z}}{\gcd(42, 60)\mathbf{Z}}$$

Proposition 11 (Properties of Tor)

1. $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$
2. $\mathrm{Tor}(\oplus_\alpha A_\alpha, B) \cong \oplus_\alpha \mathrm{Tor}(A_\alpha, B)$
3. $\mathrm{Tor}(A, B) = 0$ *if* A *or* B *is free or torsion free.*
4. $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(A_{\mathrm{tor}}, B)$ *where* A_{tor} *is the torsion subgroup of* A.
5. $\mathrm{Tor}(\mathbf{Z}_n, A) \cong \ker\left(A \xrightarrow{\times n} A\right)$
6. *The short exact sequence*

$$0 \to B \to C \to D \to 0$$

yeilds a natural exact sequence

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \\ \to A \otimes B \to A \otimes C \to A \otimes D \to 0$$

Proof of Proposition 11.

2. $\mathrm{Tor}(\oplus_\alpha A_\alpha, B) \cong \oplus_\alpha \mathrm{Tor}(A_\alpha, B)$
   - Let F_α be a free resolution of A_α
   - Then $\oplus_\alpha F_\alpha$ is a free resolution of A_α
   - $$\begin{aligned} \mathrm{Tor}(\oplus_\alpha A_\alpha, B) &\cong H_1((\oplus_\alpha F_\alpha) \otimes B) \\ &\cong H_1(\oplus_\alpha (F_\alpha \otimes B)) \\ &\cong \oplus_\alpha H_1(F_\alpha \otimes B) \\ &\cong \oplus_\alpha \mathrm{Tor}(A_\alpha, B) \end{aligned}$$

5. $\mathrm{Tor}(\mathbf{Z}_n, A) \cong \ker\left(A \xrightarrow{\times n} A\right)$
   - Use the free resolution of $\mathbf{Z}_n$
     $$\cdots \to 0 \to \mathbf{Z} \xrightarrow{\times n} \mathbf{Z} \to \mathbf{Z}_n \to 0$$
   - Tensoring with A simplifies to
     $$\cdots \to 0 \to A \xrightarrow{(\times n)} A \to 0$$

Proof of Proposition 11 *(continued)*.

3. $\text{Tor}(A, B) = 0$ if A or B is free (we will address torsion free later)
   - Suppose A is free.
   - Use the free resolution of A

$$\cdots \to 0 \to 0 \to A \to A \to 0$$

   - $\text{Tor}(A, B) = \ker(0 \to A \otimes B) = 0$
   - Suppose B is $\mathbf{Z}$
   - Then tensoring an exact free resolution $0 \to F_1 \to F_0 \to A \to 0$ with B remains exact
   - Suppose $B \cong \oplus_\alpha \mathbf{Z}$
   - Then tensoring an exact free resolution $0 \to F_1 \to F_0 \to A \to 0$ with B is a direct sum of exact sequences which is exact.

Proof of Proposition 11 *(continued)*.

6. The short exact sequence

$$0 \to B \to C \to D \to 0$$

yeilds a natural exact sequence

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \\ \to A \otimes B \to A \otimes C \to A \otimes D \to 0$$

- Choose a free resolution F of the form $0 \to F_1 \to F_0 \to A \to 0$
- All the terms of F are free so tensoring $0 \to B \to C \to D \to 0$ with F_n remains exact.
- Get short exact sequence of chain complexes

$$0 \to (F \otimes B) \to (F \otimes C) \to (F \otimes D) \to 0$$

- Apply Snake Lemma to get natural exact sequence above.

Proof of Proposition 11 *(continued)*.

1. $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$
   - Consider the six term exact sequence from part 6 coming from the short exact
     $$0 \to F_1 \to F_0 \to B \to 0$$
   - $$0 \to \mathrm{Tor}(A,F_1) \to \mathrm{Tor}(A,F_0) \to \mathrm{Tor}(A,B) \\ \to A \otimes F_1 \to A \otimes F_0 \to A \otimes B \to 0$$
   - F_0 and F_1 are free so by part 3 $\mathrm{Tor}(A,F_1) \cong \mathrm{Tor}(A,F_0) \cong 0$
   - $$0 \to \mathrm{Tor}(A,B) \to A \otimes F_1 \to A \otimes F_0 \to A \otimes B \to 0$$
   - $$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Tor}(A,B) & \longrightarrow & A \otimes F_1 & \longrightarrow & A \otimes F_0 & \longrightarrow & A \otimes B & \longrightarrow 0 \\ & & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\ 0 \longrightarrow & \mathrm{Tor}(B,A) & \longrightarrow & F_1 \otimes A & \longrightarrow & F_0 \otimes A & \longrightarrow & B \otimes A & \longrightarrow 0 \end{array}$$

Proof of Proposition 11 *(continued)*.

1. $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$ *(continued)*
   - $$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Tor}(A,B) & \xrightarrow{\alpha} & A \otimes F_1 & \xrightarrow{\beta} & A \otimes F_0 & \longrightarrow & A \otimes B & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \gamma} & & {\scriptstyle \tau}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle \mu}\Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & \mathrm{Tor}(B,A) & \xrightarrow{\alpha'} & F_1 \otimes A & \xrightarrow{\beta'} & F_0 \otimes A & \longrightarrow & B \otimes A & \longrightarrow & 0 \end{array}$$
   - We will define a homorphism $\gamma : \mathrm{Tor}(A, B) \to \mathrm{Tor}(B, A)$ preserving commutativity
   - Let $x \in \mathrm{Tor}(A, B)$
   - Claim: $\tau\alpha(x) \in \operatorname{Im}\alpha'$
   - By commutativity $\beta'\tau\alpha(x) = \mu\beta\alpha(x) = \mu(0) = 0$
   - So $\tau\alpha(x) \in \ker\beta' = \operatorname{Im}\alpha'$
   - By injectivity of α' there is a unique $x' \in \mathrm{Tor}(B, A)$ with $\alpha'(x') = \tau\alpha(x)$
   - Set $\gamma(x) = x'$.
   - γ takes 0 to 0 and sums to sums so it is a homomorphism.
   - $$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Tor}(A,B) & \longrightarrow & A \otimes F_1 & \longrightarrow & A \otimes F_0 \\ \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \gamma} & & {\scriptstyle \tau}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle \mu}\Big\downarrow{\scriptstyle \cong} \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Tor}(B,A) & \longrightarrow & F_1 \otimes A & \longrightarrow & F_0 \otimes A \end{array}$$

Proof of Proposition 11 *(continued)*.

1. $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$ *(continued)*
   - Add some trivial groups and homomorphisms to get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Tor}(A, B) & \longrightarrow & A \otimes F_1 & \longrightarrow & A \otimes F_0 \\
\Big\downarrow \cong & & \Big\downarrow \cong & & \Big\downarrow \gamma & & \tau \Big\downarrow \cong & & \mu \Big\downarrow \cong \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Tor}(B, A) & \longrightarrow & F_1 \otimes A & \longrightarrow & F_0 \otimes A
\end{array}$$

   - Now apply Five Lemma to show $\gamma : \mathrm{Tor}(A, B) \to \mathrm{Tor}(B, A)$ is an isomorphism.

Proof of Proposition 11 *(continued)*.

3. $\operatorname{Tor}(A, B) \cong 0$ if A or B is torsion free.
    - Applying part 1 assume B is torsion free.
    - Let
    $$0 \to F_1 \xrightarrow{\partial_1} F_0 \to A$$
    be a free resolution of A.
    - Get exact
    $$0 \to \operatorname{Tor}(A, B) \to F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B$$
    - Claim $\partial_1 \otimes \mathbf{1}_B$ is injective.
    - Suppose $\sum_i f_i \otimes b_i \in \ker \partial_1 \otimes \mathbf{1}_B$
    - Then then $\sum_i (\partial_1 f_i) \otimes b_i = 0$ in $F_0 \otimes B$.
    - Hence in $\mathbf{Z}[F_0 \times B]$
    $$\sum_i (\partial_1 f_i, b_i) = \sum_j (f_j^0, b_j^1 + b_j^2) - (f_j^0, b_j^1) - (f_j^0, b_j^2) + \sum_k (f_k^1 + f_k^2, b_k^0) - (f_k^1, b_k^0) - (f_k^2, b_k^2)$$
    - Let $B_0 \subset B$ be the subgroup generated by the finite set $\{b_i, b_k^0, b_j^1, b_j^2\}$

Proof of Proposition 11 *(continued)*.

3. $\text{Tor}(A, B) \cong 0$ if A or B is torsion free. *(continued)*
   - Hence in $\mathbf{Z}[F_0 \times B_0]$

$$\sum_i (\partial_1 f_i, b_i) = \sum_j (f_j^0, b_j^1 + b_j^2) - (f_j^0, b_j^1) - (f_j^0, b_j^2)$$
$$+ \sum_k (f_k^1 + f_k^2, b_k^0) - (f_k^1, b_k^0) - (f_k^2, b_k^2)$$

   - Therefore in $F_0 \otimes B_0$

$$\sum_i (\partial_1 f_i) \otimes b_i = 0$$

   - Let $B_0 \subset B$ is torsion free and finitely generated so free abelian.
   - Hence

$$F_1 \otimes B_0 \xrightarrow{\partial_1 \otimes \mathbf{1}_{B_0}} F_0 \otimes B_0$$

     is injective.
   - Thus in $F_1 \otimes B_0$

$$\sum_i f_i \otimes b_i = 0$$

   -

Proof of Proposition 11 *(continued)*.

3. $\mathrm{Tor}(A, B) \cong 0$ if A or B is torsion free. *(continued)*
   - Hence in $\mathbf{Z}[F_1 \times B_0]$

$$\sum_i (f_i, b_i) = \sum_n (f_n^3, b_n^4 + b_n^5) - (f_n^3, b_n^4) - (f_n^3, b_n^5) + \sum_k (f_m^4 + f_k^5, b_m^3) - (f_m^4, b_m^3) - (f_m^4, b_m^3)$$

   - This equality holds in $\mathbf{Z}[F_1 \times B]$
   - Thus in $F_1 \otimes B$

$$\sum_i f_i \otimes b_i = 0$$

   - It follows that $\mathrm{Tor}(A, B) = \ker(\partial_1 \otimes \mathbf{1}_B) = 0$

Proof of Proposition 11 *(continued)*.

4. $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(A_{\mathrm{tor}}, B)$ where A_{tor} is the torsion subgroup of A.
   - We have the exact sequence $0 \to A_{\mathrm{tor}} \to A \to (A/A_{\mathrm{tor}}) \to 0$
   - Apply part 6 to get exact

$$0 \to \mathrm{Tor}(B, A_{\mathrm{tor}}) \to \mathrm{Tor}(B, A) \to \mathrm{Tor}(B, A/A_{\mathrm{tor}}) \to B \otimes A_{\mathrm{tor}} \to B \otimes A \to B \otimes A/A_{\mathrm{tor}} \to 0$$

   - A/A_{tor} is torsion free so by part 3 $\mathrm{Tor}(B, A/A_{\mathrm{tor}}) = 0$
   - Get exact

$$0 \to \mathrm{Tor}(B, A_{\mathrm{tor}}) \to \mathrm{Tor}(B, A) \to 0$$

   - Thus $\mathrm{Tor}(B, A_{\mathrm{tor}}) \cong \mathrm{Tor}(B, A)$
   - and by part 1 $\mathrm{Tor}(A_{\mathrm{tor}}, B) \cong \mathrm{Tor}(A, B)$

□