Math 6802
Algebraic Topology II

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Universal coefficient theorem

Theorem 1 (Universal coefficient theorem)

If C a chain complex with each C_n free abelian there is a split short exact sequence

$$0 \to H_n(C) \otimes G \to H_n(C; G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0.$$

In particular

$$H_n(C; G) \cong \big(H_n(C) \otimes G\big) \oplus \mathrm{Tor}(H_{n-1}(C), G)$$

The short exact sequence is natural in that if $C \to D$ is a chain map then we get commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_n(C) \otimes G & \longrightarrow & H_n(C; G) & \longrightarrow & \mathrm{Tor}(H_{n-1}(C), G) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_n(D) \otimes G & \longrightarrow & H_n(D; G) & \longrightarrow & \mathrm{Tor}(H_{n-1}(D), G) & \longrightarrow & 0 \end{array}$$

Proof of Theorem 1.

- Let C be a chain complex with each C_n free abelain
- Let $Z_n = \ker \partial_n \subset C_n$ be the n-cycles
- Let $B_n = \operatorname{Im} \partial_{n+1} \subset C_n$ be the n-boundaries
- View Z and B as chain complexes.
- Z_n and B_n are subgroups of free abelian groups and hence free abelian.
- Thus we have exact

$$0 \to Z_n \to C_n \to B_{n-1} \to 0$$

- View above as a free resolution of B_{n-1}
- Get exact

$$0 \to \operatorname{Tor}(B_{n-1}, G) \to Z_n \otimes G \to C_n \otimes G \to B_{n-1} \otimes G \to 0$$

- B_{n-1} is free so $\operatorname{Tor}(B_{n-1}, G) = 0$

Proof of Theorem 1 *(continued)*.

- B_{n-1} is free so $\text{Tor}(B_{n-1}, G) = 0$
- Thus we have exact

$$0 \to Z_n \otimes G \to C_n \otimes G \to B_{n-1} \otimes G \to 0$$

- And hence exact sequence of chain complexes

$$0 \to Z \otimes G \to C \otimes G \to B \otimes G \to 0$$

- Apply Snake Lemma and fact that $\partial = 0$ for Z and B to get exact

$$\cdots \to B_n \otimes G \xrightarrow{\kappa} Z_n \otimes G \to H_n(C \otimes G) \to B_{n-1} \otimes G \xrightarrow{\kappa} Z_{n-1} \otimes G \to \cdots$$

where κ is the connecting homomorphism

Proof of Theorem 1 *(continued)*.

- What is κ?
- We have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_n \otimes G & \longrightarrow & C_n \otimes G & \xrightarrow{\partial_n \otimes \mathbf{1}} & B_{n-1} \otimes G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \partial_n \otimes \mathbf{1}} & & \downarrow & & \\
0 & \longrightarrow & Z_{n-1} \otimes G & \longrightarrow & C_{n-1} \otimes G & \xrightarrow{\partial_{n-1} \otimes \mathbf{1}} & B_{n-2} \otimes G & \longrightarrow & 0
\end{array}$$

- Given $b_{n-1} \otimes g \in B_{n-1} \otimes G$
- There is $c_n \in C_n$ such that $\partial_n c_n = b_{n-1}$
- Maps to $b_{n-1} \otimes g$ now viewed as a element of $C_{n-1} \otimes G$
- Which is the image of the element $b_{n-1} \otimes g \in Z_{n-1} \otimes G$
- Thus the connecting homomorphism is $\kappa = i_{n-1} \otimes \mathbf{1}$ where

$$i_{n-1} : B_{n-1} \to Z_{n-1}$$

is inclusion

Proof of Theorem 1 *(continued)*.

- From long exact sequence

$$\cdots \to B_n \otimes G \xrightarrow{i_n \otimes \mathbf{1}} Z_n \otimes G \to H_n(C \otimes G) \to B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes \mathbf{1}} Z_{n-1} \otimes G \to \cdots$$

- Get short exact sequence

$$0 \to \frac{Z_n \otimes G}{\mathrm{Im}(i_n \otimes \mathbf{1})} \to H_n(C \otimes G) \to \ker(i_{n-1} \otimes \mathbf{1}) \to 0$$

- By the definition of $H_{n-1}(C)$ we have a free resolution

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0$$

- Tensoring with G we get exact

$$0 \to \mathrm{Tor}(H_n(C), G) \to B_n \otimes G \xrightarrow{i_n \otimes \mathbf{1}} Z_n \otimes G \to H_n(C) \otimes G \to 0$$

Proof of Theorem 1 *(continued)*.

- Apply exactness of

$$0 \to \mathrm{Tor}(H_n(C), G) \to B_n \otimes G \xrightarrow{i_n \otimes \mathbf{1}} Z_n \otimes G \to H_n(C) \otimes G \to 0$$

- To short exact sequence

$$0 \to \frac{Z_n \otimes G}{\mathrm{Im}(i_n \otimes \mathbf{1})} \to H_n(C \otimes G) \to \ker(i_{n-1} \otimes \mathbf{1}) \to 0$$

- We see that

$$\frac{Z_n \otimes G}{\mathrm{Im}(i_n \otimes \mathbf{1})} \cong H_n(C) \otimes G$$

- and

$$\ker(i_{n-1} \otimes \mathbf{1}) \cong \mathrm{Tor}(H_n(C), G)$$

- Conclude with exactness of

$$0 \to H_n(C) \otimes G \to H_n(C; G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0.$$

□

Cellular Homology

Definition 2 (Chain complex for cellular homology)

Let X be a CW complex (see Hatcher pg. 5). Let $X^{-1} = \varnothing$ and

$$C_n^{\mathrm{cw}}(X) = \begin{cases} H_n(X^n, X^{n-1}), & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Let

$$\partial^{\mathrm{cw}} : C_n^{\mathrm{cw}}(X) \to C_{n-1}^{\mathrm{cw}}(X)$$

be the connecting homomorphism

$$H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}, X^{n-2})$$

from the long exact sequence of the triple (X^n, X^{n-1}, X^{n-2})

Proposition 3

$C^{\mathrm{cw}}(X)$ *is a chain complex*

Proof of Proposition 3.

- We must verify that $(\partial^{\mathrm{CW}})^2 = 0$.
- We have the following commutative diagram of chain maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(X^n) & \longrightarrow & C(X^{n+1}) & \longrightarrow & C(X^{n+1}, X^n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j_\sharp} & & \downarrow & & \downarrow{\scriptstyle \mathbf{1}} & & \\ 0 & \longrightarrow & C(X^n, X^{n-1}) & \longrightarrow & C(X^{n+1}, X^{n-1}) & \longrightarrow & C(X^{n+1}, X^n) & \longrightarrow & 0 \end{array}$$

- Naturality of the connecting homomorphism says the following square from the long exact sequence of homology commutes:

$$\begin{array}{ccc} H_{n+1}(X^{n+1}, X^n) & \xrightarrow{\partial} & H_n(X^n) \\ \downarrow{\scriptstyle \mathbf{1}} & & \downarrow{\scriptstyle j_*} \\ H_{n+1}(X^{n+1}, X^n) & \xrightarrow{\partial^{\mathrm{CW}}} & H_n(X^n, X^{n-1}) \end{array}$$

- So $\partial^{\mathrm{CW}} = j_* \partial$

Proof of Proposition 3 *(continued)*.

- We have commutative

$$
\begin{array}{ccccc}
 & & H_n(X^n) & & \\
 & \overset{\partial}{\nearrow} & \big\downarrow j_* & & \\
H_{n+1}(X^{n+1}, X^n) & \xrightarrow{\partial^{\mathrm{CW}}} & H_n(X^n, X^{n-1}) & \xrightarrow{\partial^{\mathrm{CW}}} & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & \big\downarrow \partial & \overset{j_*}{\nearrow} & \\
 & & H_{n-1}(X^{n-1}) & &
\end{array}
$$

- Vertical composition

$$H_n(X^n) \xrightarrow{j_*} H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1})$$

  is part of long exact sequence of the pair (X^n, X^{n-1})

- So

$$\left(\partial^{\mathrm{CW}}\right)^2 = (j_*\partial)(j_*\partial) = j_*(\partial j_*)\partial = j_* \circ 0 \circ \partial = 0$$

□

Definition 4 (Cellular Homology)

If X is a CW complex then the ***n*th cellular homology group** of X is

$$H_n^{\mathrm{CW}}(X) = H_n(C^{\mathrm{CW}}(X))$$

For a CW complex X how do $H_n^{\mathrm{CW}}(X)$ and $H_n(X)$ relate?

Theorem 5 (Cellular and sing. simp. homology agree)

If X is a CW complex then

$$H_n^{\mathrm{CW}}(X) \cong H_n(X)$$

Advantages of H^{CW}

1. If X is finite then $C^{\mathrm{CW}}(X)$ is finitely generated
2. boundary maps in $C^{\mathrm{CW}}(X)$ can be easily understood.

The proof of Theorem 5 will use the following properties of homology for CW complexes.

Lemma 6 (Homology of CW complexes)

Let X be a CW complex. Let Γ^n be the set of n-cells of X

1. $$H_k(X^n, X^{n-1}) \cong \begin{cases} \mathbf{Z}[\Gamma^n] & k = n \\ 0, & k \neq n \end{cases}$$
2. *If $k > n$*
$$H_k(X^n) = 0$$
3. *If $k < n$ and $i : X^n \to X$ is the inclusion map then*
$$H_k(X^n) \xrightarrow[i_*]{\cong} H_k(X)$$