Math 6802
Algebraic Topology II

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Definition 1 (Cellular Homology)

If X is a CW complex then the ***n*th cellular homology group** of X is

$$H_n^{\mathrm{CW}}(X) = H_n(C^{\mathrm{CW}}(X))$$

For a CW complex X how do $H_n^{\mathrm{CW}}(X)$ and $H_n(X)$ relate?

Theorem 2 (Cellular and sing. simp. homology agree)

If X is a CW complex then

$$H_n^{\mathrm{CW}}(X) \cong H_n(X)$$

Advantages of H^{CW}

1. If X is finite then $C^{\mathrm{CW}}(X)$ is finitely generated
2. boundary maps in $C^{\mathrm{CW}}(X)$ can be easily understood.

The proof of Theorem 2 will use the following properties of homology for CW complexes.

Lemma 3 (Homology of CW complexes)

Let X be a CW complex. Let Γ^n be the set of n-cells of X

1. $$H_k(X^n, X^{n-1}) \cong \begin{cases} \mathbf{Z}[\Gamma^n] & k = n \\ 0, & k \neq n \end{cases}$$
2. *If $k > n$*
$$H_k(X^n) = 0$$
3. *If $k < n$ and $i : X^n \to X$ is the inclusion map then*
$$H_k(X^n) \xrightarrow[i_*]{\cong} H_k(X)$$

Proof of Lemma 3 Claim 1.

- We have a commuting quotient maps of good pairs

$$\begin{tikzcd}(\coprod_{\alpha\in\Gamma^n} D^n, \coprod_{\alpha\in\Gamma^n} S^{n-1}) \arrow[r, "r"] \arrow[rr, bend left, "p"] & (X^n, X^{n-1}) \arrow[r, "q"] & (X^n/X^{n-1}, \{*\})\end{tikzcd}$$

- By fact that $H_n(Y,A) \cong \tilde{H}_n(Y/A)$ we get that

$$p_* : H_k\left(\coprod_{\alpha\in\Gamma^n} D^n, \coprod_{\alpha\in\Gamma^n} S^{n-1}\right) \to \tilde{H}_k(X^n/X^{n-1})$$

  and

$$q_* : H_k(X^n, X^{n-1}) \to \tilde{H}_k(X^n/X^{n-1})$$

  are isomorphisms.

- So we get an isomorphism

$$r_* : H_k\left(\coprod_{\alpha\in\Gamma^n} D^n, \coprod_{\alpha\in\Gamma^n} S^{n-1}\right) \to H_k(X^n, X^{n-1})$$

Proof of Lemma 3 Claim 1 *(continued)*.

- Hence

$$\begin{aligned}
H_k(X^n, X^{n-1}) &\cong H_k\left(X^n/X^{n+1}\right) \\
&\cong H_k\left(\coprod_{\alpha\in\Gamma^n} D^n, \coprod_{\alpha\in\Gamma^n} S^{n-1}\right) \\
&\cong \bigoplus_{\alpha\in\Gamma^n} H_k(D^n, S^{n-1}) \quad \text{(Additivity)} \\
&\cong \bigoplus_{\alpha\in\Gamma^n} \tilde{H}_k(S^n) \\
&\cong \begin{cases} \bigoplus_{\alpha\in\Gamma^n} \mathbf{Z} & k = n \\ 0, & k \neq n \end{cases} \\
&\cong \begin{cases} \mathbf{Z}[\Gamma^n] & k = n \\ 0, & k \neq n \end{cases}
\end{aligned}$$

$\square$

Proof of Lemma 3 Claim 2.

Now we show Claim 2: If $k > n$

$$H_k(X^n) = 0$$

Assume $k > n$

- In the long exact sequence for the pair (X^n, X^{n-1}) we have

$$\overbrace{H_{k+1}(X^n, X^{n-1})}^{0} \to H_k(X^{n-1}) \to H_k(X^n) \to \overbrace{H_k(X^n, X^{n-1})}^{0}$$

- So

$$H_k(X^n) \cong H_k(X^{n-1})$$

- $k > n-1 > n-2 > \cdots > 1$ so

$$H_k(X^n) \cong H_k(X^{n-1}) \cong \cdots \cong H_k(X^0) \cong 0$$

□

Proof of Lemma 3 Claim 3.

Now we show Claim 3: If $k < n$

$$H_k(X^n) = H_k(X)$$

Assume $k < n$

- Similarly, long exact sequence for the pair (X^{n+1}, X^n) gives

$$\overbrace{H_{k+1}(X^{n+1}, X^n)}^{0} \to H_k(X^n) \to H_k(X^{n+1}) \to \overbrace{H_k(X^{n+1}, X^n)}^{0}$$

- So

$$H_k(X^n) \cong H_k(X^{n+1})$$

- $k < n < n+1 < n+2 < \cdots$ so

$$H_k(X^n) \cong H_k(X^{n+1}) \cong H_k(X^{n+2}) \cong \cdots$$

- **IF** X finite dimensional then $X = X^{n+m}$ for some m and we get

$$H_k(X^n) \cong H_k(X^{n+m}) \cong H_k(X)$$

Proof of Lemma 3 Claim 3 *(continued)*.

- Claim: $H_k(X^n) \xrightarrow{i_*} H_k(X)$ is surjective
  - Let $[c] \in H_k(X)$ be a cycle. Where $c = \sum_{i=1}^{\ell} \sigma_i^k$
  - $\bigcup_{i=1}^{\ell} \sigma_i^k(\Delta^k)$ is compact so by Hatcher Prop A.1 there is some m such that
$$\bigcup_{i=1}^{\ell} \sigma_i^k(\Delta^k) \subset X^{n+m}$$
  - Thus $[c]$ is in the image of $H_k(X^{n+m}) \xrightarrow{i'_*} H_k(X)$
  - Thus $[c]$ is in the image of the composition
$$H_k(X^n) \cong H_k(X^{n+m}) \xrightarrow{i'_*} H_k(X)$$
- Claim: $H_k(X^n) \xrightarrow{i_*} H_k(X)$ is injective
  - Suppose $[c] = 0$ in $H_k(X)$
  - Then $c = \partial d$ for some chain $d \in C_{k+1}(X)$
  - There is m s.t. $d \in C_{k+1}(X^{n+m})$ so $[c] = 0$ in $H_k(X^{n+m}) \cong H_k(X^n)$

$\square$

Proof of Theorem 2.

$$
\begin{tikzcd}
 & & & & 0 \\
 & & 0 \arrow[d] & H_n(X^{n+1}) \cong H_n(X) \arrow[ur] & \\
 & & H_n(X^n) \arrow[ur] \arrow[d, "j_n"] & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial^{\mathrm{CW}}_{n+1}"'] \arrow[ur, "\partial_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[r, "\partial^{\mathrm{CW}}_n"] \arrow[d, "\partial_n"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & & \\
 & 0 \arrow[ur] & & &
\end{tikzcd}
$$

- $H_n(X) \cong \frac{H_n(X^n)}{\partial_{n+1} H_{n+1}(X^{n+1}, X^n)}$
- $\partial^{\mathrm{CW}}_{n+1} H_{n+1}(X^{n+1}, X^n) = j_n \partial_{n+1} H_{n+1}(X^{n+1}, X^n) \cong \partial_{n+1} H_{n+1}(X^{n+1}, X^n)$
- $\ker \partial^{\mathrm{CW}}_n = \ker \partial_n = j_n H_n(X^n) \cong H_n(X^n)$
- $$H_n(X) \cong \frac{H_n(X^n)}{\partial_{n+1} H_{n+1}(X^{n+1}, X^n)} \xrightarrow[j_n]{\cong} \frac{\ker \partial^{\mathrm{CW}}_n}{\partial^{\mathrm{CW}}_{n+1} H_n(X^n, X^{n-1})} = H^{\mathrm{CW}}_n(X)$$

$\square$

Degree and local degree

Definition 4 (Degree)

Let $f : S^n \to S^n$ be a map where $n > 0$. Then

$$f_* : H_n(S^n) \to H_n(S^n)$$

is a homomorphism of $\mathbf{Z}$ to $\mathbf{Z}$ which must be of the form

$$f_*(\alpha) = d \cdot \alpha$$

for some $d \in \mathbf{Z}$. The **degree** of f is

$$\deg f = d$$

- The identity map $\mathbf{1} : S^n \to S^n$ has degree 1 by functoriality of H_*
- $\deg(f \circ g) = (\deg f) \cdot (\deg g)$ by functoriality of H_*
- If f has an inverse (or just a homotopy inverse) then $\deg f = \pm 1$

Definition 5 (Local degree)

Suppose

- $f : S^n \to S^n$ is a map sending $x \in S^n$ to $y \in S^n$
- x has a neighborhood U such that $y \notin f(U - x)$
- Let $V = f(U)$.

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "\cong"'] \arrow[rrrrr, bend left, "\varphi_x"] & H_n(S^n, S^n - x) \arrow[r, "\text{exc.}", "\cong"'] & H_n(U, U - x) \arrow[r, "f_*"] & H_n(V, V - y) & H_n(S^n, S^n - y) \arrow[l, "\text{exc.}"', "\cong"] & H_n(S^n) \arrow[l, "\cong"]
\end{tikzcd}$$

There must be some $d_x \in \mathbf{Z}$ such that

$$\varphi_x(\alpha) = d_x \cdot \alpha$$

The **local degree** of f at x is

$$\deg_x f = d_x$$

- If f is a local homeomorphism near x then $\deg_x f = \pm 1$

- The following proposition gives an effective method of determining the degree of many maps $f : S^n \to S^n$

Proposition 6 (Degree from local degree)

Given $f : S^n \to S^n$ if there is $y \in S^n$ such that

$$f^{-1}(y) = \{x_1, \cdots, x_m\}$$

is a finite set then

$$\deg f = \sum_{i=1}^{m} \deg_{x_i} f$$

Example 7 (Selfmaps of S^1)

- Let $S^1 = \left\{ z \in \mathbf{C} \middle| |z| = 1 \right\}$
- Let $f_n : S^1 \to S^1$ be the map $f(z) = z^n$
- Claim: $\deg f_n = n$
- $f^{-1}(1) = \left\{ z_k = e^{\frac{2k\pi i}{n}} \in \mathbf{C} \middle| |z| = 1 \right\}$
- May homotope f so that in a neighborhood of each z_k f is a rotation.
- $\deg_{z_k} f = 1$
- $\deg f = \sum_{k=1}^{n} \deg_{z_k} f = \sum_{k=1}^{n} 1 = n$

Proposition 8

Let $Sf : S^{n+1} \to S^{n+1}$ be the suspension of $f : S^n \to S^n$. Then

$$\deg Sf = \deg f$$

Degree and the cellular boundary map

X a CW complex. Recall

- $$\Gamma^n = \{ \ n\text{-cells of } X \ \}$$
- $$C_n^{\mathrm{CW}}(X) = H_n(X^n, X^{n-1}) \cong \mathbf{Z}[\Gamma^n]$$
- $$\partial^{\mathrm{CW}} = H_k(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

  Is the connecting homomorphism of the triple (X^n, X^{n-1}, X^{n-2})

Proposition 9 (Cellular boundary formula)

- *X CW complex*
- *Let $\{e_\alpha^n\}$ be the n-cells of X*
- *Let $\left\{e_\beta^{n-1}\right\}$ be the $(n-1)$-cells of X*
- *Let $d_{\alpha\beta} = \deg \varphi_{\alpha\beta}$ where*

$$\varphi_{\alpha\beta} : S_\alpha^{n-1} \to S_\beta^{n-1}$$

is the composition

$$S_\alpha^{n-1} = \partial D_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \to \frac{X^{n-1}}{X^{n-1} - e_\beta^{n-1}} = S_\beta^{n-1}$$

Then

$$\partial^{\mathrm{CW}}(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$$