Math 6802
Algebraic Topology II

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Definition 1 (Cellular Homology)

If $X$ is a CW complex then the $n$th cellular homology group of $X$ is

$$H_{n}^{\text{CW}}(X) = H_{n}(C_{n}^{\text{CW}}(X))$$

For a CW complex $X$ how do $H_{n}^{\text{CW}}(X)$ and $H_{n}(X)$ relate?

Theorem 2 (Cellular and sing. simp. homology agree)

If $X$ is a CW complex then

$$H_{n}^{\text{CW}}(X) \cong H_{n}(X)$$

Advantages of $H_{n}^{\text{CW}}$

1. If $X$ is finite then $C_{n}^{\text{CW}}(X)$ is finitely generated
2. boundary maps in $C_{n}^{\text{CW}}(X)$ can be easily understood.
The proof of Theorem 2 will use the following properties of homology for CW complexes.

**Lemma 3 (Homology of CW complexes)**

Let $X$ be a CW complex. Let $\Gamma^n$ be the set of $n$-cells of $X$

1. 
   
   
   \[
   H_k(X^n, X^{n-1}) \cong \begin{cases} 
   \mathbb{Z}[\Gamma^n] & k = n \\
   0, & k \neq n
   \end{cases}
   \]

2. If $k > n$
   
   
   \[
   H_k(X^n) = 0
   \]

3. If $k < n$ and $i : X^n \to X$ is the inclusion map then
   
   \[
   H_k(X^n) \xrightarrow{i_*} H_k(X)
   \]
Proof of Lemma 3 Claim 1.

- We have a commuting quotient maps of good pairs

\[
(\coprod_{\alpha \in \Gamma_n} D^n, \coprod_{\alpha \in \Gamma_n} S^{n-1}) \xrightarrow{r} (X^n, X^{n-1}) \xrightarrow{q} (X^n/X^{n-1}, \{\ast\})
\]

- By fact that $H_n(Y, A) \cong \tilde{H}_n(Y/A)$ we get that

\[
p_* : H_k\left(\coprod_{\alpha \in \Gamma_n} D^n, \coprod_{\alpha \in \Gamma_n} S^{n-1}\right) \to \tilde{H}_k(X^n/X^{n-1})
\]

and

\[
q_* : H_k(X^n, X^{n-1}) \to \tilde{H}_k(X^n/X^{n-1})
\]

are isomorphisms.

- So we get an isomorphism

\[
r_* : H_k\left(\coprod_{\alpha \in \Gamma_n} D^n, \coprod_{\alpha \in \Gamma_n} S^{n-1}\right) \to H_k(X^n, X^{n-1})
\]
Proof of Lemma 3 Claim 1 (continued).

Hence

\[ H_k(X^n, X^{n-1}) \cong H_k \left( \frac{X^n}{X^{n+1}} \right) \]

\[ \cong H_k \left( \bigsqcup_{\alpha \in \Gamma^n} D^n, \bigsqcup_{\alpha \in \Gamma^n} S^{n-1} \right) \]

\[ \cong \bigoplus_{\alpha \in \Gamma^n} H_k(D^n, S^{n-1}) \quad \text{(Additivity)} \]

\[ \cong \bigoplus_{\alpha \in \Gamma^n} \tilde{H}_k(S^n) \]

\[ \cong \begin{cases} \bigoplus_{\alpha \in \Gamma^n} \mathbb{Z} & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases} \]

\[ \cong \begin{cases} \mathbb{Z}[\Gamma^n] & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases} \]
Proof of Lemma 3 Claim 2.

Now we show Claim 2: If \( k > n \)

\[
H_k(X^n) = 0
\]

Assume \( k > n \)

- In the long exact sequence for the pair \((X^n, X^{n-1})\) we have

\[
\begin{array}{c}
0 \\
H_{k+1}(X^n, X^{n-1}) \rightarrow H_k(X^{n-1}) \rightarrow H_k(X^n) \rightarrow H_k(X^n, X^{n-1}) \\
0
\end{array}
\]

- So

\[
H_k(X^n) \cong H_k(X^{n-1})
\]

- \( k > n - 1 > n - 2 > \cdots > 1 \) so

\[
H_k(X^n) \cong H_k(X^{n-1}) \cong \cdots \cong H_k(X^0) \cong 0
\]
Proof of Lemma 3 Claim 3.

Now we show Claim 3: If $k < n$

\[ H_k(X^n) = H_k(X) \]

Assume $k < n$

- Similarly, long exact sequence for the pair $(X^{n+1}, X^n)$ gives

\[
\begin{array}{ccc}
0 & \rightarrow & H_{k+1}(X^{n+1}, X^n) \\
& \rightarrow & H_k(X^n) \\
& \rightarrow & H_k(X^{n+1}) \\
& \rightarrow & 0
\end{array}
\]

- So

\[ H_k(X^n) \cong H_k(X^{n+1}) \]

- $k < n < n + 1 < n + 2 < \cdots$ so

\[ H_k(X^n) \cong H_k(X^{n+1}) \cong H_k(X^{n+2}) \cong \cdots \]

- **IF** $X$ finite dimensional then $X = X^{n+m}$ for some $m$ and we get

\[ H_k(X^n) \cong H_k(X^{n+m}) \cong H_k(X) \]
Proof of Lemma 3 Claim 3 (continued).

Claim: $H_k(X^n) \xrightarrow{i_*} H_k(X)$ is surjective

- Let $[c] \in H_k(X)$ be a cycle. Where $c = \sum_{i=1}^\ell \sigma_i^k$
- $\bigcup_{i=1}^\ell \sigma_i^k(\Delta^k)$ is compact so by Hatcher Prop A.1 there is some $m$ such that
  \[
  \bigcup_{i=1}^\ell \sigma_i^k(\Delta^k) \subset X^{n+m}
  \]
- Thus $[c]$ is in the image of $H_k(X^{n+m}) \xrightarrow{i'_*} H_k(X)$
- Thus $[c]$ is in the image of the composition
  \[
  H_k(X^n) \cong H_k(X^{n+m}) \xrightarrow{i'_*} H_k(X)
  \]

Claim: $H_k(X^n) \xrightarrow{i_*} H_k(X)$ is injective

- Suppose $[c] = 0$ in $H_k(X)$
- Then $c = \partial d$ for some chain $d \in C_{k+1}(X)$
- There is $m$ s.t. $d \in C_{k+1}(X^{n+m})$ so $[c] = 0$ in $H_k(X^{n+m}) \cong H_k(X^n)$
Proof of Theorem 2.

\[ H_n(X) \cong \frac{H_n(X^n)}{\partial_{n+1} H_{n+1}(X^{n+1}, X^n)} \]

\[ \partial_{n+1} H_{n+1}(X^{n+1}, X^n) = j_n \partial_{n+1} H_{n+1}(X^{n+1}, X^n) \cong \partial_{n+1} H_{n+1}(X^{n+1}, X^n) \]

\[ \ker \partial_n^{CW} = \ker \partial_n = j_n H_n(X^n) \cong H_n(X^n) \]

\[ H_n(X) \cong \frac{H_n(X^n)}{\partial_{n+1} H_{n+1}(X^{n+1}, X^n)} \xrightarrow{j_n} \frac{\ker \partial_n^{CW}}{\partial_{n+1} H_n(X^n, X^{n-1})} = H_n^{CW}(X) \]
Definition 4 (Degree)

Let \( f : S^n \to S^n \) be a map where \( n > 0 \). Then

\[
f_* : H_n(S^n) \to H_n(S^n)
\]

is a homomorphism of \( \mathbb{Z} \) to \( \mathbb{Z} \) which must be of the form

\[
f_*(\alpha) = d \cdot \alpha
\]

for some \( d \in \mathbb{Z} \). The degree of \( f \) is

\[
\text{deg } f = d
\]

- The identity map \( 1 : S^n \to S^n \) has degree 1 by functoriality of \( H_* \). 
- \( \text{deg}(f \circ g) = (\text{deg } f) \cdot (\text{deg } g) \) by functoriality of \( H_* \). 
- If \( f \) has an inverse (or just a homotopy inverse) then \( \text{deg } f = \pm 1 \).
Definition 5 (Local degree)

Suppose

- $f : S^n \to S^n$ is a map sending $x \in S^n$ to $y \in S^n$
- $x$ has a neighborhood $U$ such that $y \notin f(U - x)$
- Let $V = f(U)$.

There must be some $d_x \in \mathbb{Z}$ such that

$$\varphi_x(\alpha) = d_x \cdot \alpha$$

The **local degree** of $f$ at $x$ is

$$\deg_x f = d_x$$

- If $f$ is a local homeomorphism near $x$ then $\deg_x f = \pm 1$
The following proposition gives an effective method of determining the degree of many maps \( f : S^n \rightarrow S^n \)

**Proposition 6 (Degree from local degree)**

*Given \( f : S^n \rightarrow S^n \) if there is \( y \in S^n \) such that \( f^{-1}(y) = \{x_1, \cdots, x_m\} \) is a finite set then

\[
\deg f = \sum_{i=1}^{m} \deg_{x_i} f
\]
Example 7 (Selfmaps of $S^1$)

- Let $S^1 = \left\{ z \in \mathbb{C} \middle| |z| = 1 \right\}$
- Let $f_n : S^1 \to S^1$ be the map $f(z) = z^n$
- Claim: $\deg f_n = n$
- $f^{-1}(1) = \left\{ z_k = e^{\frac{2k\pi i}{n}} \in \mathbb{C} \middle| |z| = 1 \right\}$
- May homotope $f$ so that in a neighborhood of each $z_k$ $f$ is a rotation.
- $\deg_{z_k} f = 1$
- $\deg f = \sum_{k=1}^{n} \deg_{z_k} f = \sum_{k=1}^{n} 1 = n$

Proposition 8

Let $Sf : S^{n+1} \to S^{n+1}$ be the suspension of $f : S^n \to S^n$. Then

$$\deg Sf = \deg f$$
Degree and the cellular boundary map

\( X \) a CW complex. Recall

\[ \Gamma^n = \{ n\text{-cells of } X \} \]

\[ C_n^{CW}(X) = H_n(X^n, X^{n-1}) \cong \mathbb{Z}[\Gamma^n] \]

\[ \partial_{CW} = H_k(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2}) \]

Is the connecting homomorphism of the triple \((X^n, X^{n-1}, X^{n-2})\)
Proposition 9 (Cellular boundary formula)

- $X$ CW complex
- Let $\{e^n_\alpha\}$ be the $n$-cells of $X$
- Let $\{e^{n-1}_\beta\}$ be the $(n - 1)$-cells of $X$
- Let $d_{\alpha\beta} = \deg \varphi_{\alpha\beta}$ where

\[ \varphi_{\alpha\beta} : S^{n-1}_\alpha \to S^{n-1}_\beta \]

is the composition

\[ S^{n-1}_\alpha = \partial D_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{X^{n-1}} \frac{X^{n-1}}{X^{n-1} - e^{n-1}_\beta} = S^{n-1}_\beta \]

Then

\[ \partial^{CW}(e^n_\alpha) = \sum_{\beta} d_{\alpha\beta} e^{n-1}_\beta \]