Math 6802
Algebraic Topology II

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Degree and the cellular boundary map

X a CW complex. Recall

- $$\Gamma^n = \{ \text{ } n\text{-cells of } X \text{ } \}$$
- $$C_n^{\mathrm{CW}}(X) = H_n(X^n, X^{n-1}) \cong \mathbf{Z}[\Gamma^n]$$
- $$\partial^{\mathrm{CW}} = H_k(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

  Is the connecting homomorphism of the triple (X^n, X^{n-1}, X^{n-2})

Proposition 1 (Cellular boundary formula)

- *X CW complex*
- *Let $\{e_\alpha^n\}$ be the n-cells of X*
- *Let $\left\{e_\beta^{n-1}\right\}$ be the $(n-1)$-cells of X*
- *Let $d_{\alpha\beta} = \deg \varphi_{\alpha\beta}$ where*

$$\varphi_{\alpha\beta} : S_\alpha^{n-1} \to S_\beta^{n-1}$$

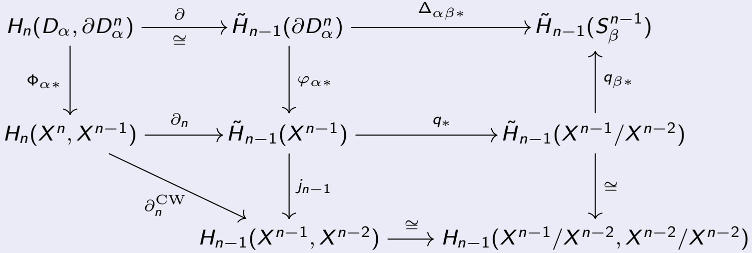
is the composition

$$S_\alpha^{n-1} = \partial D_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \to \frac{X^{n-1}}{X^{n-1} - e_\beta^{n-1}} = S_\beta^{n-1}$$

Then

$$\partial^{\mathrm{CW}}(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$$

Proof of Proposition 1.



- $\Phi_\alpha : D_\alpha \to X^n$ the characteristic map
- $\varphi_\alpha : \partial D_\alpha \to X^{n-1}$ the attaching map
- $q : X^{n-1} \to X^{n-1}/X^{n-2}$ the quotient map
- $q_\beta : X^{n-1}/X^{n-2} \to S_\beta^{n-1}$ the quotient map
- $\Delta_{\alpha\beta} = q_\beta q \varphi_\alpha$
- upper left and lower left commutative by naturality. Lower right comm. at chain map level. Upper right comm. at quotient space level.

Proof of Proposition 1 *(continued)*.

$$
\begin{array}{ccccc}
H_n(D_\alpha, \partial D_\alpha^n) & \xrightarrow[\cong]{\partial} & \tilde{H}_{n-1}(\partial D_\alpha^n) & \xrightarrow{\Delta_{\alpha\beta *}} & \tilde{H}_{n-1}(S_\beta^{n-1}) \\
\downarrow{\scriptstyle \Phi_{\alpha *}} & & \downarrow{\scriptstyle \varphi_{\alpha *}} & & \uparrow{\scriptstyle q_{\beta *}} \\
H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & \tilde{H}_{n-1}(X^{n-1}) & \xrightarrow{q_*} & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \\
& \searrow{\scriptstyle \partial_n^{\mathrm{CW}}} & \downarrow{\scriptstyle j_{n-1}} & & \downarrow{\scriptstyle \cong} \\
& & H_{n-1}(X^{n-1}, X^{n-2}) & \xrightarrow{\cong} & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{array}
$$

- $\tilde{H}_{n-1}(X^{n-1}/X^{n-2}) = \bigoplus_\beta q_{\beta *}^{-1}(\tilde{H}_{n-1}(S_\beta^{n-1}))$
- Let $1_\beta \in \tilde{H}_{n-1}(S_\beta^{n-1})$ be the generator.
- $d_{\alpha\beta} \cdot 1_\beta = (\deg \Delta_{\alpha\beta}) \cdot 1_\beta = q_{\beta *} q_* \varphi_{\alpha *} \partial(1)$
- $\partial^{\mathrm{CW}} e_\alpha^n = q_* \varphi_{\alpha *} \partial(1) = \sum_\beta q_{\beta *}^{-1} q_{\beta *} q_* \varphi_{\alpha *} \partial(1) =$
  $\sum_\beta q_{\beta *}^{-1}(d_{\alpha\beta} \cdot 1_\beta) = \sum_\beta d_{\alpha\beta} q_{\beta *}^{-1}(1_\beta) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$

□

Classical invariants

Definition 2 (Torsion and rank)

For an abelian group A the **torsion subgroup** is

$$A_{\mathrm{tor}} = \left\{ a \in A \middle| \exists n \in \mathbf{Z} \text{ s.t. } na = 0 \right\}$$

There is a unique cardinal n such that

$$A/A_{\mathrm{tor}} \cong \oplus^n \mathbf{Z}$$

The **rank** of A is

$$\operatorname{rank} A = n$$

Lemma 3 (Rank-nullity Theorem)

$B \subset A$ abelian groups then

$$\operatorname{rank} B + \operatorname{rank} A/B = \operatorname{rank} A$$

Definition 4 (Betti number)

For a space X the k**th Betti number** is

$$b_k(X) = \operatorname{rank} H_k(X)$$

Definition 5 (Euler characteristic)

If $H(X)$ has finite rank the **Euler characteristic** is

$$\chi(X) = \sum_k (-1)^k b_k(X)$$

- We see immediately that Euler characteristic and Betti numbers are invariants of homotopy type
- $b_0(X)$ is the number of path components of X

Example 6 (Invariants of spheres)

The Betti numbers for S^n are

$$b_k(S^n) = \begin{cases} 1, & k \in \{0, n\} \\ 0, & k \notin \{0, n\} \end{cases}$$

The Euler characteristic for S^n is

$$\chi(S^n) = 1 + (-1)^n = \begin{cases} 2, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Definition 7 (Finite CW complex)

A CW complex is **finite** if it has finitely many cells.

Proposition 8 (Euler characteristic of a CW complex)

Given a finite CW complex $X = X^n$ let α_k be the number of k-cells of X.

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \alpha_k$$

Proposition 9

If X and Y are finite CW complexes then

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y)$$