Math 6802
Algebraic Topology II

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January 30, 2015

The Hom$(-, B)$ functor

Definition 1 (Opposite category)

If $\mathscr{C}$ is a category then the **opposite category** $\mathscr{C}^{\mathrm{op}}$ is the category with objects

$$\mathrm{Ob}(\mathscr{C}) = \mathrm{Ob}(\mathscr{C}^{\mathrm{op}})$$

and for $C_1, C_2 \in \mathscr{C}$ morphisms

$$\mathrm{Mor}_{\mathscr{C}^{\mathrm{op}}}(C_1, C_2) = \mathrm{Mor}_{\mathscr{C}}(C_2, C_1).$$

The composition rule for morphisms being $m_2 \circ_{\mathscr{C}^{\mathrm{op}}} m_1 = m_1 \circ_{\mathscr{C}} m_2$.

Definition 2

A **contravariant functor** $F : \mathscr{C} \to \mathscr{D}$ is a (covariant) functor $F : \mathscr{C} \to \mathscr{D}^{\mathrm{op}}$.

Example 3 ($- \otimes B$ is covariant)

Fix the abelian group B. The functor $F : \mathbf{Ab} \to \mathbf{Ab}$ with $F(A) = A \otimes B$ and $F(h) = h \otimes \mathbf{1}_B$ is covariant.

Example 4 ($\mathrm{Hom}(-, B)$ is contravariant)

Fix the abelian group B. The functor $G : \mathbf{Ab} \to \mathbf{Ab}$ with $G(A) = \mathrm{Hom}(A, B)$ and $G(h)(f) = f \circ h$ is contravariant.

Example 5 ($\mathrm{Hom}(A, -)$ is covariant)

Fix the abelian group A. The functor $G : \mathbf{Ab} \to \mathbf{Ab}$ with $G(B) = \mathrm{Hom}(A, B)$ and $G(h)(f) = h \circ f$ is covariant.

- We now can now repeat Lectures 3-5 with $\mathrm{Hom}(-, B)$ instead of $- \otimes B$.
- When B is fixed we will call $A^* = \mathrm{Hom}(A, B)$ the **dual** of A
- and given $f : C \to D$ we get its **dual** $f^* : D^* \to C^*$ where $f^*(h) = h \circ f$.

Proposition 6 (Properties of Hom)

1. $$\mathrm{Hom}(\oplus_\alpha A_\alpha, B) \cong \prod_\alpha \mathrm{Hom}(A_\alpha, B)$$
2. $$\mathrm{Hom}(A, \oplus_\beta B_\beta) \cong \bigoplus_\beta \mathrm{Hom}(A, B_\beta)$$
3. $$\mathrm{Hom}(\mathbf{Z}, B) \cong B$$

Hom and exactness

Proposition 7 (Left exactness for $\mathrm{Hom}(-, B)$)

Fix the abelian group B. Given exact

$$C \xrightarrow{\alpha} D \xrightarrow{\beta} E \to 0$$

then after dualizing with B we get exact

$$C^* \xleftarrow{\alpha^*} D^* \xleftarrow{\beta^*} E^* \leftarrow 0.$$

Thus $\mathrm{Hom}(-, B)$ *is left exact (this is the definition for contravariant functors)*

$$\mathrm{Hom}(A, C) \xrightarrow{\alpha \circ -} \mathrm{Hom}(A, D) \xrightarrow{\beta \circ -} \mathrm{Hom}(A, E) \to 0$$

is exact (thus $\mathrm{Hom}(A, -)$ *is right exact).*

1. Prove the left and right exactness statements above.
2. Show that the opposite exactness statements do not hold in general.

Hom$(-, B)$ and chain complexes

- Recall that a chain complex (C_*, ∂) is a graded abelian group

$$C_* = \bigoplus_{n \in \mathbf{Z}} C_n$$

together with a degree -1 homomorrphism $\partial : C_* \to C_*$ satisfying $\partial^2 = 0$.

Definition 8 (Cochain complex)

Fix abelian group B. The **cochain complex** (C^*, δ) corresponding to the chain complex (C_*, ∂) is the abelian group

$$C^* = \mathrm{Hom}(C_*, B)$$

together with the **coboundary map** $\delta = \partial^*$

- Note that

$$C^* = \mathrm{Hom}(C_*, B) = \mathrm{Hom}\left(\bigoplus_n C_n, B\right) = \prod_{n\in\mathbf{Z}} \mathrm{Hom}(C_n, B) = \prod_{n\in\mathbf{Z}} C_n^*$$

but we will generally only use the subgroup

$$\bigoplus_{n\in\mathbf{Z}} C_n^* \subset \prod_{n\in\mathbf{Z}} C_n^*.$$

- Also note that δ is degree 1 but we do have $\delta^2 = - \circ \partial^2 = 0$.
- In order to preserve the convention that the domain of f_n is C_n we set

$$\delta_n = \delta|_{C_n^*} = \partial_{n+1}^*$$

Definition 9 (Cohomology groups)

Given a cochain complex (C^*, δ) its n**th cohomology group** is

$$H^n(C^*) = \frac{\ker \delta_n}{\mathrm{Im}\, \delta_{n-1}}.$$

Lemma 10

If

$$f : C \to D$$

is a chain map then

$$f^* : D^* \to C^*$$

is a cochain map.

Proof.

$f : C \to D$ is a chain map so $\partial_D f = f \partial_C$

$$\begin{aligned} f^* \circ \delta_D &= f^*(- \circ \partial_D) \\ &= (- \circ \partial_D) \circ f \\ &= - \circ f \circ \partial_C \\ &= \delta_C \circ (- \circ f) \\ &= \delta_C \circ f^* \end{aligned}$$

□

Lemma 11

If

$$T : C \to D$$

is a chain homotopy between chain maps $f, g : C \to D$ *then*

$$T^* : D^* \to C^*$$

is a cochain homotopy between chain maps f^* *and* g^**. Note: cochain homotopies have degree -1.*

Proof.

$T : C \to D$ is a chain homotopy so $\partial T + T\partial = g - f$

$$\begin{aligned}
\delta T^* + T^*\delta &= \delta \circ (- \circ T) + T^* \circ (- \circ \partial) \\
&= (- \circ T) \circ \partial + (- \circ \partial) \circ T \\
&= - \circ (T\partial + \partial T) \\
&= - \circ (g - f) \\
&= (g - f)^* \\
&= g^* - f^*
\end{aligned}$$

□

Cohomology with coefficients in G

Definition 12 (Cohomology with coefficients in G)

Fix the abelian group G for dualizing. Given a chain complex C let

$$H^n(C; G) = H^n(C^*)$$

If X is a space let

$$H^n(X; G) = H^n(C^*(X))$$

$$\tilde{H}^n(X; G) = H^n(\tilde{C}^*(X))$$

$$H^n_{\mathrm{CW}}(X; G) = H^n(C^{\mathrm{CW}*}(X))$$

For (X, A) a topological pair let

$$H^n(X, A; G) = H^n(C^*(X, A))$$

Example 13 (Cohomology of $\mathbf{R}\mathrm{P}^4$)

Standard CW chain complex for $\mathbf{R}\mathrm{P}^4$ is

$$\cdots \to 0 \to \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{\times 0} \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{\times 0} \mathbf{Z} \to 0$$

Dualizing (with coefficient group $G = \mathbf{Z}$) we get $C^{\mathrm{CW}*}(\mathbf{R}\mathrm{P}^4)$

$$\cdots \leftarrow 0 \leftarrow \mathbf{Z} \xleftarrow{\times 2} \mathbf{Z} \xleftarrow{\times 0} \mathbf{Z} \xleftarrow{\times 2} \mathbf{Z} \xleftarrow{\times 0} \mathbf{Z} \leftarrow 0$$

So

$$H^n_{\mathrm{CW}}(\mathbf{R}\mathrm{P}^4; \mathbf{Z}) \cong \begin{cases} \mathbf{Z}, & n = 0 \\ \mathbf{Z}_2, & n \in \{2, 4\} \\ 0, & \text{otherwise} \end{cases}$$

Homology with coefficients in $\mathbf{Z}$ are also universal for cohomology with coefficients in G for all abelian groups G.

Theorem 14 (Universal coefficient theorem for cohomology)

For C a chain complex there is a split short exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C; G) \to \mathrm{Hom}(H_n(C), G) \to 0.$$

This sequence is natural in that if $C \to D$ is a chain map then we get commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(D), G) & \longrightarrow & H^n(D; G) & \longrightarrow & \mathrm{Hom}(H_n(D), G) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(C), G) & \longrightarrow & H^n(C; G) & \longrightarrow & \mathrm{Hom}(H_n(C), G) & \longrightarrow & 0 \end{array}$$

The Ext functor

Definition 15 (The Ext functor)

Let A and B be abelian groups and F be a free resolution of A.

Dualizing F with B we get the cochain complex F^*

$$\mathrm{Ext}^n(A, B) = H^n(F^*)$$

We showed that free resolutions of the same group are chain homotopic and the dual of a chain homotopy is a cochain homotopy. Thus we have:

Theorem 16 (Ext is well-defined)

$\mathrm{Ext}^n(A, B)$ *is independent of the free resolution F of A.*

- Every abelian group A has a free resolution

$$\cdots \to 0 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \to 0$$

- So we have the chain complex F

$$\cdots \to 0 \to F_1 \xrightarrow{\partial_1} F_0 \to 0$$

- and cochain complex F^* from dualizing with B

$$\cdots \to 0 \leftarrow F_1^* \xleftarrow{\delta_0} F_0^* \leftarrow 0$$

- $$\mathrm{Ext}^n(A, B) = H^n(F^*) = \begin{cases} \ker \delta_0, & n = 0 \\ \operatorname{coker} \delta_0, & n = 1 \\ 0, & n \neq 0, 1 \end{cases}$$

- recall given $h : M \to N$ we set $\operatorname{coker} h = N/h(M)$

- We have the exact sequence

$$F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \to 0$$

- Which dualizes to exact

$$F_1^* \xleftarrow{\delta_0} F_0^* \leftarrow A^* \leftarrow 0$$

- Which we may extend to exact

$$0 \leftarrow \operatorname{coker} \delta_0 \leftarrow F_1^* \xleftarrow{\delta_0} F_0^* \leftarrow A^* \leftarrow 0$$

- Hence

$$\operatorname{Ext}^0(A, B) \cong \ker \delta_0 \cong A^* = \operatorname{Hom}(A, B)$$

- and $\operatorname{Ext}^1(A, B)$ fits in the exact sequence

$$0 \leftarrow \operatorname{Ext}^1(A, B) \leftarrow F_1^* \xleftarrow{\delta_0} F_0^* \leftarrow \operatorname{Hom}(A, B) \leftarrow 0$$

Definition 17

$$\operatorname{Ext}(A, B) = \operatorname{Ext}^1(A, B)$$

Example 18

Let's compute $\mathrm{Ext}(\mathbf{Z}_{60}, \mathbf{Z}_{42})$

- Free resolution F of $\mathbf{Z}_{60}$

$$\cdots \to 0 \to \mathbf{Z} \xrightarrow{\times 60} \mathbf{Z} \to 0$$

- Dualizing by $\mathbf{Z}_{42}$ we get F^*

$$0 \leftarrow \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}_{42}) \xleftarrow{\times 60} \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}_{42}) \leftarrow 0$$

- Simplifying

$$0 \leftarrow \mathbf{Z}_{42} \xleftarrow{\times 60} \mathbf{Z}_{42} \leftarrow 0$$

- Hence

$$\mathrm{Ext}(\mathbf{Z}_{60}, \mathbf{Z}_{42}) \cong \mathrm{coker}(\times 60) \cong \frac{\mathbf{Z}_{42}}{60\mathbf{Z}_{42}} = \frac{\mathbf{Z}}{\gcd(42, 60)\mathbf{Z}}$$

Proposition 19 (Properties of Ext)

1. $\mathrm{Ext}(\oplus_\alpha A_\alpha, B) \cong \prod_\alpha \mathrm{Ext}(A_\alpha, B)$
2. $\mathrm{Ext}(A, \oplus_\beta B_\beta) \cong \bigoplus_\beta \mathrm{Ext}(A, B_\beta)$
3. $\mathrm{Ext}(A, B) = 0$ *if A is free.*
4. $\mathrm{Ext}(\mathbf{Z}_n, B) \cong B/nB$

Proof of Proposition 19.

1. Claim: $\operatorname{Ext}(\oplus_\alpha A_\alpha, B) \cong \prod_\alpha \operatorname{Ext}(A_\alpha, B)$
   - Let F_α be a free resolution of A_α
   - Then $\oplus_\alpha F_\alpha$ is a free resolution of $\oplus_\alpha A_\alpha$
   - $$\begin{aligned}\operatorname{Ext}(\oplus_\alpha A_\alpha, B) &\cong H^1((\oplus_\alpha F_\alpha)^*) \\ &\cong H^1(\Pi_\alpha F_\alpha^*) \\ &\cong \prod_\alpha H^1(F_\alpha^*) \\ &\cong \prod_\alpha \operatorname{Ext}(A_\alpha, B)\end{aligned}$$

Proof of Proposition 19.

2. Claim: $\mathrm{Ext}(A, \oplus_\beta B_\beta) \cong \bigoplus_\beta \mathrm{Ext}(A, B_\beta)$
   - Let F be a free resolution of A
   - $$\begin{aligned}\mathrm{Ext}(A, \oplus_\beta B_\beta) &\cong H^1(\mathrm{Hom}(F, \oplus_\beta B_\beta)) \\ &\cong H^1(\oplus_\beta \mathrm{Hom}(F, B_\beta)) \\ &\cong \bigoplus_\beta H^1(\mathrm{Hom}(F, B_\beta)) \\ &\cong \bigoplus_\beta \mathrm{Ext}(A, B_\beta)\end{aligned}$$

Proof of Proposition 19 *(continued)*.

3. Claim: $\operatorname{Ext}(A, B) = 0$ if A is free
   - Suppose A is free.
   - Use the free resolution of A

$$\cdots \to 0 \to A \to 0$$

   - $\operatorname{Ext}(A, B) = \operatorname{coker}(A \to 0) = 0$.
4. Claim: $\operatorname{Ext}(\mathbf{Z}_n, B) \cong B/nB$
   - Use the free resolution of $\mathbf{Z}_n$

$$\cdots \to 0 \to \mathbf{Z} \xrightarrow{\times n} \mathbf{Z} \to 0$$

   - Get $\operatorname{Ext}(A, B) \cong \operatorname{coker}\left(B \xrightarrow{\times n} B\right) = B/nB$.

□

Universal coefficient theorem for cohomology

Theorem 20 (Universal coefficient theorem for cohomology)

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This sequence is natural in that if $C \to D$ is a chain map then we get commutative

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