Math 6802
Algebraic Topology II

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Universal coefficient theorem for cohomology

Theorem 1 (Universal coefficient theorem for cohomology)

For C a chain complex there is a split short exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C; G) \to \mathrm{Hom}(H_n(C), G) \to 0.$$

This sequence is natural in that if $C \to D$ is a chain map then we get commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(D), G) & \longrightarrow & H^n(D; G) & \longrightarrow & \mathrm{Hom}(H_n(D), G) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(C), G) & \longrightarrow & H^n(C; G) & \longrightarrow & \mathrm{Hom}(H_n(C), G) & \longrightarrow & 0 \end{array}$$

Proof of Theorem 1.

- Let C be a chain complex with each C_n free abelain
- Let $Z_n = \ker \partial_n \subset C_n$ be the n-cycles
- Let $B_n = \operatorname{Im} \partial_{n+1} \subset C_n$ be the n-boundaries
- View Z and B as chain complexes.
- Z_n and B_n are subgroups of free abelian groups and hence free abelian.
- Thus we have exact

$$0 \to Z_n \to C_n \to B_{n-1} \to 0$$

- View above as a free resolution of B_{n-1}
- Get exact

$$0 \leftarrow \operatorname{Ext}(B_{n-1}, G) \leftarrow Z_n^* \leftarrow C_n^* \leftarrow B_{n-1}^* \leftarrow 0$$

- B_{n-1} is free so $\operatorname{Ext}(B_{n-1}, G) = 0$

Proof of Theorem 1 *(continued)*.

- Thus we have exact

$$0 \leftarrow Z_n^* \leftarrow C_n^* \leftarrow B_{n-1}^* \leftarrow 0$$

- And hence exact sequence of cochain complexes (shift degree of B^* by 1)

$$0 \leftarrow Z^* \leftarrow C^* \leftarrow B^* \leftarrow 0$$

- Apply Snake Lemma and fact that $\delta = 0$ for Z^* and B^* to get exact

$$\cdots \leftarrow B_n^* \xleftarrow{\kappa} Z_n^* \leftarrow H^n(C^*) \leftarrow B_{n-1}^* \xleftarrow{\kappa} Z_{n-1}^* \leftarrow \cdots$$

  where κ is the connecting homomorphism

Proof of Theorem 1 *(continued)*.

- From long exact sequence

$$\cdots \leftarrow B_n^* \xleftarrow{i_n^*} Z_n^* \leftarrow H^n(C^*) \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow \cdots$$

- Get short exact sequence

$$0 \leftarrow \ker i_n^* \leftarrow H^n(C^*) \leftarrow \operatorname{coker} i_{n-1}^* \leftarrow 0$$

We just need to recognize this as desired s.e.s.

□