Math 6802
Algebraic Topology II

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Product spaces and the Künneth Formula

- Suppose X and Y are spaces
- How does $H(X \times Y)$ relate to $H(X)$ and $H(Y)$?
- $H_0(X \times Y) = H_0(X) \otimes H_0(Y)$
- If X and Y are path connected then

$$H_1(X \times Y) = \pi_1(X \times Y)_{\mathrm{ab}} = (\pi_1(X) \times \pi_1(Y))_{\mathrm{ab}} = H_1(X) \oplus H_1(Y)$$

- Künneth formula gives the precise connection

Theorem 1 (The Künneth Formula)

If X and Y are spaces then there is a natural exact sequence

$$0 \to \bigoplus_i H_i(X) \otimes H_{n-i}(Y) \to H_n(X \times Y) \to \bigoplus_i \operatorname{Tor}(H_i(X), H_{n-i-1}(Y)) \to 0$$

This sequence splits (unnaturally). In particular

$$H_n(X \times Y) \cong \bigoplus_i H_i(X) \otimes H_{n-i}(Y) \oplus \operatorname{Tor}(H_i(X), H_{n-i-1}(Y))$$

Compare to the Universal Coefficient Theorem.

Example 2 (Homology of T^n)

- $T^1 = S^1$
- Inductively define $T^{n+1} = S^1 \times T^n$
- Claim: $H_k(T^n) = \mathbf{Z}^{\binom{n}{k}}$
- $$H_k(T^1) = H_k(S^1) = \begin{cases} \mathbf{Z}, & k = 0,1 \\ 0, & k \neq 0,1 \end{cases} = \mathbf{Z}^{\binom{1}{k}}$$
- $$\begin{aligned} H_k(T^{n+1}) &= H_k(S^1 \times T^n) \\ &= \bigoplus_i H_i(S^1) \otimes H_{k-i}(T^n) \oplus \mathrm{Tor}(H_i(S^1), H_{k-i-1}(T^n)) \\ &\cong \bigoplus_i H_i(S^1) \otimes H_{k-i}(T^n) \\ &\cong H_0(S^1) \otimes H_k(T^n) \oplus H_1(S^1) \otimes H_{k-1}(T^n) \\ &\cong H_k(T^n) \oplus H_{k-1}(T^n) \\ &\cong \mathbf{Z}^{\binom{n}{k}} \oplus \mathbf{Z}^{\binom{n}{k-1}} \\ &\cong \mathbf{Z}^{\binom{n+1}{k}} \end{aligned}$$

Proof of the Künneth Formula follows from two results:

Theorem 3 (Eilenberg-Zilber Theorem)

For spaces X and Y there is a chain homotopy equivalence

$$C(X \times Y) \simeq C(X) \otimes C(Y)$$

Theorem 4 (The Künneth Formula for Chain Complexes)

For free abelian chain complexes C and D there is a natural short exact sequence

$$0 \to \bigoplus_i H_i(C) \otimes H_{n-i}(D) \to H_n(C \otimes D) \to \bigoplus_i \mathrm{Tor}(H_i(C), H_{n-i-1}(D)) \to 0$$

We must define the chain complex structure for $C \otimes D$

Definition 5 (Tensor product of Chain Complexes)

Given chain complexes C and D the tensor product $C \otimes D$ is a chain complex with

$$(C \otimes D)_n = \bigoplus_i C_i \otimes D_{n-i}$$

and for $c \in C_i$ and $d \in D_{n-i}$

$$\partial(c \otimes d) = (\partial c) \otimes d + (-1)^i c \otimes (\partial d)$$

We check that $\partial^2 = 0$

$$\begin{aligned}
\partial^2(c \otimes d) &= \partial((\partial c) \otimes d + (-1)^i c \otimes (\partial d)) \\
&= (\partial^2 c) \otimes d + (-1)^{i-1}(\partial c) \otimes (\partial d) \\
&\qquad + (-1)^i(\partial c) \otimes (\partial d) + c \otimes (\partial^2 d) \\
&= 0
\end{aligned}$$

- Let D be the chain complex with

$$D_k = \begin{cases} A, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

  Then

$$C \otimes D = C \otimes A$$

- Künneth Formula for chain complexes becomes the Universal Coefficient Theorem
- We will start with the proof of the Künneth Formula for chain complexes.

Proposition 6

If

$$f : A \to B$$

$$g : C \to D$$

are chain maps a then

$$f \otimes g : A \otimes C \to B \otimes D$$

is a chain map.

Proof of Theorem 4 (Künneth for chain complexes).

- Let C and D be free chain complexes
- Consider the case where $\partial = 0$ for C
  - Then $\partial(c \otimes d) = (-1)^i c \otimes \partial d$
  - And $H_i(C) = C_i$
  - Hence
    $$(C \otimes D)_n = \bigoplus_i C_i \otimes D_{n-i}$$
  - So $C \otimes D$ is (as a chain complex) a direct sum of shifted copies of D
  - $$H_n(C \otimes D) = \bigoplus_i C_i \otimes H_{n-i}(D) = \bigoplus_i H_i(C) \otimes H_{n-i}(D)$$
  - $C_i = H_i(C)$ is free so $\mathrm{Tor}(H_i(C), H_{n-i-1}(D)) = 0$
  - Hence Theorem holds in this case ($\partial = 0$ on C)

Proof of Theorem 4.

- Let $B = \operatorname{Im} \partial$ be the sub chain complex of C
- Let $Z = \ker \partial$ be the sub chain complex of C
- As with the proof of the Universal Coefficient Theorem we have exact
$$0 \to Z_i \to C_i \xrightarrow{\partial} B_{i-1} \to 0$$
- And hence exact sequence of free chain complexes
$$0 \to Z \to C \xrightarrow{\partial} B \to 0$$
- Tensoring with (free) D we get exact
$$0 \to Z \otimes D \to C \otimes D \xrightarrow{\partial \otimes \mathbf{1}_D} B \otimes D \to 0$$
- From the Snake Lemma we get the long exact sequence
$$\cdots \to H_n(B \otimes D) \xrightarrow{\kappa} H_n(Z \otimes D) \to H_n(C \otimes D)$$
$$\to H_{n-1}(B \otimes D) \xrightarrow{\kappa} H_{n-1}(Z \otimes D) \to \cdots$$

Proof of Theorem 4.

- Let B and Z have $\partial = 0$ so

$$H_n(Z \otimes D) = \bigoplus_i Z_i \otimes H_{n-i}(D)$$

$$H_{n-1}(B \otimes D) = \bigoplus_i B_i \otimes H_{n-i-1}(D)$$

- Our exact sequence

$$\cdots \to H_n(B \otimes D) \xrightarrow{\kappa} H_n(Z \otimes D) \to H_n(C \otimes D) \to H_{n-1}(B \otimes D) \xrightarrow{\kappa} H_{n-1}(Z \otimes D) \to \cdots$$

- becomes

$$\cdots \to \bigoplus_i B_i \otimes H_{n-i}(D) \xrightarrow{\kappa_n} \bigoplus_i Z_i \otimes H_{n-i}(D) \to H_n(C \otimes D) \to \bigoplus_i B_i \otimes H_{n-i-1}(D) \xrightarrow{\kappa_{n-1}} \bigoplus_i Z_i \otimes H_{n-i-1}(D) \to \cdots$$

Proof of Theorem 4.

- We have the free resolution

$$0 \to B_i \to Z_i \to H_i(C) \to 0$$

- Tensoring with $H_{n-i}(D)$ gives the exact sequence

$$\begin{aligned} 0 \to \operatorname{Tor}(H_i(C), H_{n-i}(D)) \to B_i \otimes H_{n-i}(D) \\ \to Z_i \otimes H_{n-i}(D) \to H_i(C) \otimes H_{n-i}(D) \to 0 \end{aligned}$$

- Long exact sequence gives short exact sequence

$$0 \to \operatorname{coker} \kappa_n \to H_n(C \otimes D) \to \ker \kappa_{n-1} \to 0$$

- where we may identify

$$\ker \kappa_{n-1} \cong \bigoplus_i \operatorname{Tor}(H_i(C), H_{n-i-1}(D))$$

- and

$$\operatorname{coker} \kappa_n \cong \bigoplus_i H_i(C) \otimes H_{n-i}(D)$$

Proof of Theorem 4.

- Yeilding the desired exact sequence

$$0 \to \bigoplus_i H_i(C) \otimes H_{n-i}(D) \to H_n(C \otimes D)$$
$$\to \bigoplus_i \mathrm{Tor}(H_i(C), H_{n-i-1}(D)) \to 0$$

- Claim: There is a spliting map

$$s : H_n(C \otimes D) \to \bigoplus_i H_i(C) \otimes H_{n-i}(D)$$

  - We split exact $0 \to Z_i \to C_i \to B_{i-1} \to 0$
  - Hence we have $s : C_i \to Z_i$
  - Have $Z_i \to H_i(C)$
  - Gives chain maps $C \to H(C)$ and $D \to H(D)$
  - Get chain map $C \otimes D \to H(C) \otimes H(D)$
  - Induced map on homology is splitting map s.

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