Product spaces and the K"unneth Formula

- Suppose $X$ and $Y$ are spaces
- How does $H(X \times Y)$ relate to $H(X)$ and $H(Y)$?
- $H_0(X \times Y) = H_0(X) \otimes H_0(Y)$
- If $X$ and $Y$ are path connected then

  $H_1(X \times Y) = \pi_1(X \times Y)_{ab} = (\pi_1(X) \times \pi_1(Y))_{ab} = H_1(X) \oplus H_1(Y)$

- K"unneth formula gives the precise connection
Theorem 1 (The K"unneth Formula)

If \( X \) and \( Y \) are spaces then there is a natural exact sequence

\[
0 \rightarrow \bigoplus_i H_i(X) \otimes H_{n-i}(Y) \rightarrow H_n(X \times Y) \\
\rightarrow \bigoplus_i \text{Tor}(H_i(X), H_{n-i-1}(Y)) \rightarrow 0
\]

This sequence splits (unnaturally). In particular

\[
H_n(X \times Y) \cong \bigoplus_i H_i(X) \otimes H_{n-i}(Y) \oplus \text{Tor}(H_i(X), H_{n-i-1}(Y))
\]

Compare to the Universal Coefficient Theorem.
Example 2 (Homology of $T^n$)

- $T^1 = S^1$
- Inductively define $T^{n+1} = S^1 \times T^n$
- Claim: $H_k(T^n) = \mathbb{Z}^{n \choose k}$

\[ H_k(T^1) = H_k(S^1) = \begin{cases} \mathbb{Z}, & k = 0, 1 \\ 0, & k \neq 0, 1 \end{cases} = \mathbb{Z}^{1 \choose k} \]

\[ H_k(T^{n+1}) = H_k(S^1 \times T^n) \]

\[ \cong \bigoplus H_i(S^1) \otimes H_{k-i}(T^n) \oplus \text{Tor}(H_i(S^1), H_{k-i-1}(T^n)) \]

\[ \cong H_0(S^1) \otimes H_k(T^n) \oplus H_1(S^1) \otimes H_{k-1}(T^n) \]

\[ \cong H_k(T^n) \oplus H_{k-1}(T^n) \]

\[ \cong \mathbb{Z}^{n \choose k} \oplus \mathbb{Z}^{n \choose k-1} \]

\[ \cong \mathbb{Z}^{n+1 \choose k} \]
Proof of the Küneth Formula follows from two results:

**Theorem 3 (Eilenberg-Zilber Theorem)**

For spaces $X$ and $Y$ there is a chain homotopy equivalence

$$C(X \times Y) \cong C(X) \otimes C(Y)$$

**Theorem 4 (The Küneth Formula for Chain Complexes)**

For free abelian chain complexes $C$ and $D$ there is a natural short exact sequence

$$0 \rightarrow \bigoplus_{i} H_i(C) \otimes H_{n-i}(D) \rightarrow H_n(C \otimes D) \rightarrow \bigoplus_{i} \text{Tor}(H_i(C), H_{n-i-1}(D)) \rightarrow 0$$
We must define the chain complex structure for $C \otimes D$

**Definition 5 (Tensor product of Chain Complexes)**

Given chain complexes $C$ and $D$ the tensor product $C \otimes D$ is a chain complex with

$$(C \otimes D)_n = \bigoplus_i C_i \otimes D_{n-i}$$

and for $c \in C_i$ and $d \in D_{n-i}$

$$\partial(c \otimes d) = (\partial c) \otimes d + (-1)^i c \otimes (\partial d)$$

We check that $\partial^2 = 0$

$$\partial^2(c \otimes d) = \partial((\partial c) \otimes d + (-1)^i c \otimes (\partial d))$$

$$= (\partial^2 c) \otimes d + (-1)^i (-1)^{i-1} (\partial c) \otimes (\partial d)$$

$$+ (-1)^i (\partial c) \otimes (\partial d) + c \otimes (\partial^2 d)$$

$$= 0$$
Let $D$ be the chain complex with

$$D_k = \begin{cases} A, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

Then

$$C \otimes D = C \otimes A$$

K"unneth Formula for chain complexes becomes the Universal Coefficient Theorem

We will start with the proof of the K"unneth Formula for chain complexes.
Proposition 6

If

\[ f : A \rightarrow B \]
\[ g : C \rightarrow D \]

are chain maps, then

\[ f \otimes g : A \otimes C \rightarrow B \otimes D \]

is a chain map.
Proof of Theorem 4 (Künneth for chain complexes).

Let $C$ and $D$ be free chain complexes.

Consider the case where $\partial = 0$ for $C$.

Then $\partial(c \otimes d) = (-1)^i c \otimes \partial d$.

And $H_i(C) = C_i$.

Hence

$$(C \otimes D)_n = \bigoplus_i C_i \otimes D_{n-i}$$

So $C \otimes D$ is (as a chain complex) a direct sum of shifted copies of $D$.

$$H_n(C \otimes D) = \bigoplus_i C_i \otimes H_{n-i}(D) = \bigoplus_i H_i(C) \otimes H_{n-i}(D)$$

$C_i = H_i(C)$ is free so $\text{Tor}(H_i(C), H_{n-i-1}(D)) = 0$.

Hence Theorem holds in this case ($\partial = 0$ on $C$).
Proof of Theorem 4.

- Let $B = \text{Im} \partial$ be the sub chain complex of $C$
- Let $Z = \ker \partial$ be the sub chain complex of $C$
- As with the proof of the Universal Coefficient Theorem we have exact
  \[ 0 \to Z_i \to C_i \xrightarrow{\partial} B_{i-1} \to 0 \]
- And hence exact sequence of free chain complexes
  \[ 0 \to Z \to C \xrightarrow{\partial} B \to 0 \]
- Tensoring with (free) $D$ we get exact
  \[ 0 \to Z \otimes D \to C \otimes D \xrightarrow{\partial \otimes 1_D} B \otimes D \to 0 \]
- From the Snake Lemma we get the long exact sequence
  \[
  \cdots \to H_n(B \otimes D) \xrightarrow{\kappa} H_n(Z \otimes D) \to H_n(C \otimes D) \\
  \to H_{n-1}(B \otimes D) \xrightarrow{\kappa} H_{n-1}(Z \otimes D) \to \cdots
  \]
Proof of Theorem 4.

Let $B$ and $Z$ have $\partial = 0$ so

$$H_n(Z \otimes D) = \bigoplus_i Z_i \otimes H_{n-i}(D)$$

$$H_{n-1}(B \otimes D) = \bigoplus_i B_i \otimes H_{n-i-1}(D)$$

Our exact sequence

$$\cdots \to H_n(B \otimes D) \xrightarrow{\kappa} H_n(Z \otimes D) \to H_n(C \otimes D)$$

$$\rightarrow H_{n-1}(B \otimes D) \xrightarrow{\kappa} H_{n-1}(Z \otimes D) \to \cdots$$

becomes

$$\cdots \to \bigoplus_i B_i \otimes H_{n-i}(D) \xrightarrow{\kappa_n} \bigoplus_i Z_i \otimes H_{n-i}(D) \to H_n(C \otimes D)$$

$$\rightarrow \bigoplus_i B_i \otimes H_{n-i-1}(D) \xrightarrow{\kappa_{n-1}} \bigoplus_i Z_i \otimes H_{n-i-1}(D) \to \cdots$$
Proof of Theorem 4.

- We have the free resolution

\[
0 \to B_i \to Z_i \to H_i(C) \to 0
\]

- Tensoring with \( H_{n-i}(D) \) gives the exact sequence

\[
0 \to \text{Tor}(H_i(C), H_{n-i}(D)) \to B_i \otimes H_{n-i}(D) \\
\to Z_i \otimes H_{n-i}(D) \to H_i(C) \otimes H_{n-i}(D) \to 0
\]

- Long exact sequence gives short exact sequence

\[
0 \to \text{coker } \kappa_n \to H_n(C \otimes D) \to \ker \kappa_{n-1} \to 0
\]

- where we may identify

\[
\ker \kappa_{n-1} \cong \bigoplus_i \text{Tor}(H_i(C), H_{n-i-1}(D))
\]

- and

\[
\text{coker } \kappa_n \cong \bigoplus_i H_i(C) \otimes H_{n-i}(D)
\]
Proof of Theorem 4.

- Yeilding the desired exact sequence

\[ 0 \rightarrow \bigoplus_i H_i(C) \otimes H_{n-i}(D) \rightarrow H_n(C \otimes D) \rightarrow \bigoplus_i \text{Tor}(H_i(C), H_{n-i-1}(D)) \rightarrow 0 \]

- Claim: There is a splitting map

\[ s : H_n(C \otimes D) \rightarrow \bigoplus_i H_i(C) \otimes H_{n-i}(D) \]

- We split exact \( 0 \rightarrow Z_i \rightarrow C_i \rightarrow B_{i-1} \rightarrow 0 \)
- Hence we have \( s : C_i \rightarrow Z_i \)
- Have \( Z_i \rightarrow H_i(C) \)
- Gives chain maps \( C \rightarrow H(C) \) and \( D \rightarrow H(D) \)
- Get chain map \( C \otimes D \rightarrow H(C) \otimes H(D) \)
- Induced map on homology is splitting map \( s \).