Math 6802
Algebraic Topology II

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Theorem 1 (The Künneth Formula)

If X and Y are spaces then there is a natural exact sequence

$$0 \to \bigoplus_i H_i(X) \otimes H_{n-i}(Y) \to H_n(X \times Y) \to \bigoplus_i \mathrm{Tor}(H_i(X), H_{n-i-1}(Y)) \to 0$$

This sequence splits (unnaturally). In particular

$$H_n(X \times Y) \cong \bigoplus_i H_i(X) \otimes H_{n-i}(Y) \oplus \mathrm{Tor}(H_i(X), H_{n-i-1}(Y))$$

Proof of the Künneth Formula follows from two results:

Theorem 2 (The Künneth Formula for Chain Complexes)

For free abelian chain complexes C and D there is a natural short exact sequence

$$0 \to \bigoplus_i H_i(C) \otimes H_{n-i}(D) \to H_n(C \otimes D) \to \bigoplus_i \mathrm{Tor}(H_i(C), H_{n-i-1}(D)) \to 0$$

Theorem 3 (Eilenberg-Zilber Theorem)

For spaces X and Y there is a chain homotopy equivalence

$$C(X) \otimes C(Y) \simeq C(X \times Y)$$

The cross product for singular simplicial chains

- To prove the Eilenberg-Zilber Theorem we need a chain map from $C(X) \otimes C(Y)$ to $C(X \times Y)$ which induces the desired chain homotopy.
- This chain map will be called the **cross product**

$$- \times - : C_p(X) \otimes C_q(Y) \to C_{p+q}(X \times Y)$$

- Given a singular p-simplex $\sigma : \Delta^p \to X$ and q-simplex $\eta : \Delta^q \to Y$
- We want $(p+q)$-chain in $X \times Y$.
- Key point is to triangulate $\Delta^p \times \Delta^q$.

Definition 4 (Poset)

A **poset** is a pair $(S, \leq)$ where S is a set and $\leq$ is a binary relation on S satisfying:

1. Reflexivity ($a \leq a$)
2. Antisymmetry ($a \leq b$ and $b \leq a$ implies $a = b$)
3. Transitivity ($a \leq b$ and $b \leq c$ implies $a \leq c$)

- Two elements of a poset $s, t \in S$ are **comparable** if $s \leq t$ or $t \leq s$.
- A subset L of a poset S is **linearly ordered** if the elements of L are pairwise comparable.

Definition 5 (Geometric realization of a poset)

If S is a poset then the **geometric realization** of S is the simplicial complex $|S|$ with

1. 0-skeleton
$$|S|^0 = S$$
2. And an n-simplex joining the vertices $s_0, s_1, \cdots, s_n \in S$ for each finite, linearly ordered subset $\{s_0, s_1, \cdots, s_n\} \subset S$.

- Note that the n-simplex Δ^n is the geometric realization of the linearly ordered poset

$$\{x_0 < x_1 < \cdots < x_n\}.$$

- We turn to the definition of the cross product

$$- \times - : C_p(X) \otimes C_q(Y) \to C_{p+q}(X \times Y)$$

- Recall our triangulation of $\Delta^n \times I = \Delta^n \times \Delta^1$ from the proof of Homotopy Axiom of homology:
  - Let $S = \{s_0 < s_1 < \cdots < s_n\}$ and $T = \{t_0 < t_1\}$. Then $\Delta^n = |S|$ and $\Delta^1 = |T|$
  - Vertex set of $\Delta^n \times \Delta^1$ is $S \times T$
  - We triangulated $\Delta^n \times \Delta^1$ with maximal simplices of form

$$|\{(s_0, t_0), (s_1, t_0), \cdots, (s_i, t_0), (s_i, t_1), \cdots (s_n, t_1)\}|$$

  - Notice that this is exactly the geometric realization of the poset $S \times T$ with the **product order** $(s, t) \leq (u, v)$ if $s \leq u$ and $t \leq v$.

- Let X and Y spaces with singular simplices $\sigma : \Delta^p \to X$ and $\eta : \Delta^q \to Y$
- Let $S = \{s_0 < s_1 < \cdots < s_p\}$ so that $\Delta^p = |S|$.
- Let $T = \{t_0 < t_1 < \cdots < t_q\}$ so that $\Delta^q = |T|$.
- We may identify each maximal linearly ordered subset

  $$L = \{l_0 = (s_0, t_0) < \cdots < l_{p+q} = (s_p, t_q)\} \subset S \times T$$

  with a path in the $p \times q$ grid moving from $(0, 0)$ to (p, q) monotonically upwards and rightwards.
- Define $\operatorname{sgn} L = (-1)^{b_L}$ where b_L is number of boxes in $p \times q$ grid below path for L.
- We may view $|L|$ as a subset of the affine space $|S \times T| = \Delta^p \times \Delta^q$

Definition 6 (Cross product of for singular simplicial chains)

The **cross product** is the chain map
$- \times - : C(X) \otimes C(Y) \to C(X \times Y)$ where

$$\sigma \times \eta = \sum_{\substack{L = \{l_0 < \cdots < l_{p+q}\} \subset S \times T \\ \text{maximal linearly ordered}}} (\operatorname{sgn} L) \sigma \times \eta|_{[l_0, \cdots, l_{p+q}]}$$

Note that we are *defining* the symbol $\times$ on the LHS and writing $\sigma \times \eta$ on the RHS to denote the product of the continuous maps σ and η.

Lemma 7

The cross product is a chain map.

Proof of Lemma 7.

$$\partial(- \times -)(\sigma \otimes \eta) = \partial(\sigma \times \eta) = \sum_{\substack{M = \{m_0 < \cdots < m_{p+q}\} \\ \text{maximal linearly} \\ \text{ordered subset} \\ \text{of } S \times T}} (\operatorname{sgn} M)\partial\left(\sigma \times \eta|_{[m_0,\cdots,m_{p+q}]}\right)$$

- For a linearly ordered set $L = \{l_0 < \cdots < l_k\} \subset S \times T$ let $\theta_L = \sigma \times \eta|_{[l_0,\cdots,l_k]}$ be the singular simplex.
- After applying ∂ we get a formal linear combination of maps of the form θ_L where $L = \{l_0 < \cdots < l_{p+q-1}\} \subset S \times T$ is one term short of being maximal.

Proof of Lemma 7*(continued)*.

- There are two possibilities for such a linearly ordered $L \subset S \times T$:
  1. L is a subset of precisely two maximal linearly ordered sets $M, M' \subset S \times T$.
  2. L is a subset of precisely one maximal linearly ordered set $M \subset S \times T$.
- If L is of the first type then $\operatorname{sgn} M = -\operatorname{sgn} M'$ and θ_L terms of $(\operatorname{sgn} M)\partial\theta_M$ and $(\operatorname{sgn} M')\partial\theta_{M'}$ cancel. Hence θ_L does not appear in $\partial(- \times -)(\sigma \otimes \eta)$.
- If L is of second type then $L = M - \{m_i\}$ for some unique maximal linearly ordered $M = \{m_0 < \cdots < m_{p+q}\}$. Set $k_L = i$.

Thus we see that

$$\partial(- \times -)(\sigma \otimes \eta) = \sum_{\substack{L = \{l_0 < \cdots < l_{p+q-1}\} \\ \exists! \text{ maximal linear } M \\ \text{with } L \subset M \subset S \times T}} (\operatorname{sgn} M)(-1)^{k_L}\theta_L$$

Proof of Lemma 7 *(continued)*.

Now apply the cross product and boundary map in the reverse order.

$$\begin{aligned}
(- \times -)\partial(\sigma \otimes \eta) &= (- \times -)(\partial\sigma \otimes \eta + (-1)^p \sigma \otimes \partial\eta) \\
&= (\partial\sigma) \times \eta + (-1)^p \sigma \times (\partial\eta) \\
&= \sum_{\substack{L = \{l_0 < \cdots < l_{p+q-1}\} \\ \textbf{maximal} \text{ linearly} \\ \text{ordered subset} \\ \text{of } (S - \{s_i\}) \times T}} (\operatorname{sgn} L)(-1)^i \theta_L \\
&\quad + \sum_{\substack{L = \{l_0 < \cdots < l_{p+q-1}\} \\ \textbf{maximal} \text{ linearly} \\ \text{ordered subset} \\ \text{of } S \times (T - \{t_i\})}} (-1)^p (\operatorname{sgn} L)(-1)^i \theta_L
\end{aligned}$$

Proof of Lemma 7 *(continued)*.

- The maximal linearly ordered sets $M \subset (S - \{s_i\}) \times T$ or $M \subset S \times (T - \{t_i\})$ in the two sums above are exactly the linearly ordered subsets of $S \times T$ which extend uniquely to a maximal linearly ordered subset of $S \times T$.
- Thus nonzero terms agree.
- It remains to show that signs agree.

□

The Alexander-Whitney map

- We wish to show that the cross product $- \times - : C(X) \otimes C(Y) \to C(X \times Y)$ induces a chain homotopy equivalence.
- It will be convenient to define its chain homotopy inverse the **Alexander-Whitney map**

$$A : C(X \times Y) \to C(X) \otimes C(Y)$$

- Let $S = \{s_0 < \cdots < s_n\}$ be the linearly ordered set and let $\Delta^n = |S|$.
- Define the **front p-face** map of the simplex Δ^n to be the inclusion

$$f_p : \Delta^p \to \Delta^n$$

  where $\Delta^p = |\{s_0, \cdots, s_p\}| \subset |S|$.
- Define the **back q-face** map of the simplex Δ^n to be the inclusion

$$b_q : \Delta^q \to \Delta^n$$

  where $\Delta^q = |\{s_{n-q}, \cdots, s_n\}| \subset |S|$.

Definition 8 (The Alexander-Whitney map)

Let $\sigma : \Delta^n \to X \times Y$ be a singular n-simplex of $X \times Y$. We define the **Alexander-Whitney map**

$$A : C(X \times Y) \to C(X) \otimes C(Y)$$

on σ with the formula:

$$A(\sigma) = \sum_{i=0}^{n} (\pi_X \circ \sigma \circ f_i) \otimes (\pi_Y \circ \sigma \circ b_{n-i}).$$

Lemma 9

The Alexander-Whitney map is a chain map.

Proof of Lemma 9.

$$\begin{aligned}
\partial A\sigma &= \partial \sum_{i=0}^{n} (\sigma \circ f_i) \otimes (\sigma \circ b_{n-i}). \\
&= \sum_{i=0}^{n} \partial \big[(\sigma \circ f_i) \otimes (\sigma \circ b_{n-i}) \big] \\
&= \sum_{i=0}^{n} (\partial(\sigma \circ f_i)) \otimes (\sigma \circ b_{n-i}) + (-1)^i (\sigma \circ f_i) \otimes (\partial(\sigma \circ b_{n-i}))\big] \\
&= \sum_{i=0}^{n} \left(\sum_{j=0}^{i} (-1)^j (\sigma \circ f_i|_{[s_0, \cdots, \widehat{s_j}, \cdots, s_i]})) \otimes (\sigma \circ b_{n-i}) \right. \\
&\qquad \left. + \sum_{j=i}^{n} (-1)^i (-1)^{j-i} (\sigma \circ f_i) \otimes (\sigma \circ b_{n-i}|_{[s_i, \cdots, \widehat{s_j}, \cdots, s_n]}) \right)
\end{aligned}$$

Proof of Lemma 9 *(continued)*.

$$
\begin{aligned}
&= \sum_{i=0}^{n} \left(\sum_{j=0}^{i} (-1)^j (\sigma \circ f_i|_{[s_0,\cdots,\widehat{s_j},\cdots,s_i]})) \otimes (\sigma \circ b_{n-i}) \right. \\
&\quad \left. + \sum_{j=i}^{n} (-1)^j (\sigma \circ f_i) \otimes (\sigma \circ b_{n-i}|_{[s_i,\cdots,\widehat{s_j},\cdots,s_n]}) \right) \\
&= \sum_{0 \le j < i \le n} (-1)^j (\sigma \circ f_i|_{[s_0,\cdots,\widehat{s_j},\cdots,s_i]})) \otimes (\sigma \circ b_{n-i}) \\
&\quad + \sum_{0 \le i < j \le n} (-1)^j (\sigma \circ f_i) \otimes (\sigma \circ b_{n-i}|_{[s_i,\cdots,\widehat{s_j},\cdots,s_n]})
\end{aligned}
$$

Proof of Lemma 9 *(continued)*.

$$
\begin{aligned}
A\partial\sigma &= A\sum_{j=0}^{n}(-1)^j \sigma|_{[s_0,\cdots,\widehat{s_j},\cdots,s_n]} \\
&= \sum_{0\le j<i\le n}(-1)^j(\sigma\circ f_i|_{[s_0,\cdots,\widehat{s_j},\cdots,s_i]}))\otimes(\sigma\circ b_{n-i}) \\
&\quad + \sum_{0\le i<j\le n}(-1)^j(\sigma\circ f_i)\otimes(\sigma\circ b_{n-i}|_{[s_i,\cdots,\widehat{s_j},\cdots,s_n]})
\end{aligned}
$$

□