Math 6802
Algebraic Topology II

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Homotopy inverse of the cross product

- We have the cross product chain map

$$- \times - : C_p(X) \otimes C_q(Y) \to C_{p+q}(X \times Y)$$

- and the Alexander-Whitney chain map

$$A : C_n(X \times Y) \to (C(X) \otimes C(Y))_n$$

- Thus we have chain maps

$$A \circ (- \times -) : C_p(X) \otimes C_q(Y) \to (C(X) \otimes C(Y))_{p+q}$$

- and

$$(- \times -) \circ A : C_n(X \times Y) \to C_n(X \times Y)$$

The acyclic models theorem

Definition 1 (Category with models)

A **category with models** is a category $\mathscr{C}$ with specified set of objects $\mathscr{M} \subset \mathscr{C}$ called the **models** of $\mathscr{C}$.

Definition 2 (Free functor with respect to the models)

Given a category $\mathscr{C}$ with models $\mathscr{M}$ a functor

$$T : \mathscr{C} \to \mathbf{Ab}$$

is **free with respect to** $\mathscr{M}$ if for each $M \in \mathscr{M}$ there is $e_M \in T(M)$ such that for all $X \in \mathrm{Ob}(\mathscr{C})$

$$\left\{ T(f)(e_M) \middle| M \in \mathscr{M}, f \in \mathrm{Mor}(M, X) \right\}$$

is a free basis for $T(X)$.

Definition 3 (Free functor with respect to the models)

Given a category $\mathscr{C}$ with models $\mathscr{M} = \cup_n \mathscr{M}_n$ a functor

$$T : \mathscr{C} \to \mathbf{Chain}$$

is **free with respect to** $\mathscr{M} = \cup_n \mathscr{M}_n$ if each T_n is free with respect to $\mathscr{M}_n$

Theorem 4 (Acyclic models theorem)

- *$\mathscr{C}$ a category with models $\mathscr{M} \subset \mathscr{C}$*
- *$S, T : \mathscr{C} \to$ **Chain** functors with $S_n = T_n = 0$ for $n < 0$*
- *T is free with respect to the models $\mathscr{M} = \cup_n \mathscr{M}_n$*
- *$H_i(S(M)) = 0$ for $i > 0$ for all models $M \in \mathscr{M}$ (M is S-**acyclic**)*
- *There is a natural transformation*

$$\Phi : H_0(T) \to H_0(S)$$

Then there is a natural transformation

$$\varphi : T \to S$$

which induces Φ unique up to chain homotopy.

Notation for product spaces

For a product space $X \times Y$ we have continuous projection maps

$$\pi_0 = \pi_X : X \times Y \to X$$

and

$$\pi_1 = \pi_Y : X \times Y \to Y$$

and given continuous $f : W \to X$ and $g : W \to Y$ let

$$f \times g : W \to X \times Y$$

be the continuous map

$$f \times g(w) = (f(w), g(w))$$

Observe that for any $u : W \to X \times Y$ we have $u = (\pi_X u) \times (\pi_Y u)$

Category of ordered pairs of spaces

Let **Top** $\times$ **Top** be the category of ordered pairs of spaces with morphisms from (X, Y) to (Z, W) given by pairs of maps (f, g) with $f : X \to Z$ and $g : Y \to W$ continuous.

Proof of Eilenberg-Zilber.

- For now assume Acyclic Models Theorem.
- We have two functors $S, T : \mathbf{Top} \times \mathbf{Top} \to \mathbf{Chain}$ where

$$S(X, Y) = C(X) \otimes C(Y) \qquad T(X, Y) = C(X \times Y)$$

- Let $T : \mathbf{Top} \times \mathbf{Top} \to \mathbf{Chain}$ be the functor

$$T(X, Y) = C(X \times Y)$$

and given a morphism $(f, g) : (X, Y) \to (Z, W)$ and a singular simplex $\sigma : \Delta^n \to X \times Y$ define

$$T(f, g) : C(X \times Y) \to C(Z \times W)$$

$$T(f, g)(\sigma) = (f\pi_X \sigma) \times (g\pi_Y \sigma)$$

- Can you show $T(f, g)$ is a chain map?

Proof of Eilenberg-Zilber *(continued).*

- Let $S : \mathbf{Top} \times \mathbf{Top} \to \mathbf{Chain}$ be the functor

$$S(X, Y) = C(X) \otimes C(Y)$$

and given a morphism $(f, g) : (X, Y) \to (Z, W)$ and a singular simplices $\sigma : \Delta^p \to X$ and $\eta : \Delta^q \to Y$ define

$$S(f, g) : C(X) \otimes C(Y) \to C(Z) \otimes C(W)$$

$$S(f, g)(\sigma \otimes \eta) = (f\sigma) \otimes (g\eta)$$

- Can you show $S(f, g)$ is a chain map?

Proof of Eilenberg-Zilber *(continued).*

- Set $M_n = (\Delta^n, \Delta^n)$
- Choose models for **Top** $\times$ **Top** the singleton sets

$$\mathscr{M}_n = \{M_n\}$$

- Choose $e_n \in T_n(M_n) = C_n(\Delta^n \times \Delta^n)$ to be the n-chain

$$\mathbf{1} \times \mathbf{1} : \Delta^n \to \Delta^n \times \Delta^n$$

This is the **diagonal map** for Δ^n. Note that

$$\pi_0 e_n = \pi_1 e_n = \mathbf{1}_{\Delta^n}$$

Proof of Eilenberg-Zilber *(continued).*

- Claim that the functor T_n is free with models $\mathscr{M}_n$
  - In other words, we must show that $T(X, Y) = C(X \times Y)$ has free basis
$$\mathscr{B} = \left\{ T_n(f, g)(e_n) \middle| (f, g) : (\Delta^n, \Delta^n) \to (X, Y) \right\}$$
  - We claim that $\mathscr{B}$ is exactly the set of singular n-simplices of $X \times Y$.
  - Given a singular simplex $\sigma : \Delta^n \to X \times Y$
$$\begin{aligned} T_n(\pi_0\sigma, \pi_1\sigma)(e_n) &= (\pi_0\sigma\pi_0 e_n) \times (\pi_1\sigma\pi_1 e_n) \\ &= (\pi_0\sigma\mathbf{1}_{\Delta^n}) \times (\pi_1\sigma\mathbf{1}_{\Delta^n}) \\ &= (\pi_0\sigma) \times (\pi_1\sigma) \\ &= \sigma \end{aligned}$$
  - Hence $\mathscr{B}$ contains the singular n-simplices of $X \times Y$.
  - $T_n(f, g)(e_n) = (f\pi_0 e_n) \times (g\pi_1 e_n) = f \times g$ is a map from Δ^n to $X \times Y$ so the set of singular simplices of $X \times Y$ contains $\mathscr{B}$.
  - Finally if $(f, g) \neq (f', g')$ then
$$T_n(f, g)(e_n) = f \times g \neq f' \times g' = T_n(f', g')(e_n)$$

Proof of Eilenberg-Zilber *(continued).*

- Claim that each M_n is S-acyclic
  - In other words, we must show that for $k > 0$ we have $H_k(S(M_n)) = 0$
  - $$\begin{aligned} H_k(S(M_n)) &= H_k(S(\Delta^n, \Delta^n)) \\ &= H_k(C(\Delta^n) \otimes C(\Delta^n)) \\ &\cong \bigoplus_i H_i(C(\Delta^n)) \otimes H_{k-i}(C(\Delta^n)) \\ &\quad \oplus \bigoplus_i \mathrm{Tor}(H_i(C(\Delta^n)), H_{k-i-1}(C(\Delta^n))) \\ &\cong \bigoplus_i H_i(\Delta^n) \otimes H_{k-i}(\Delta^n) \\ &\quad \oplus \bigoplus_i \mathrm{Tor}(H_i(\Delta^n), H_{k-i-1}(\Delta^n)) \\ &= \begin{cases} 0, & k \neq 0 \\ \mathbf{Z}, & k = 0 \end{cases} \end{aligned}$$

Proof of Eilenberg-Zilber *(continued).*

- Claim there is a natural isomorphism $\Phi : H_0(T) \to H_0(S)$
- In other words for each (X, Y) we have an isomorphism

$$\Phi_{(X,Y)} : H_0(T(X, Y)) \to H_0(S(X, Y))$$

such that for any morphism $(f, g) : (X, Y) \to (Z, W)$ we get commutative

$$\begin{array}{ccc} H_0(T(X,Y)) & \xrightarrow{\Phi_{(X,Y)}} & H_0(S(X,Y)) \\ \downarrow{\scriptstyle H_0T(f,g)} & & \downarrow{\scriptstyle H_0S(f,g)} \\ H_0(T(Z,W)) & \xrightarrow{\Phi_{(Z,W)}} & H_0(S(Z,W)) \end{array}$$

- Set $\Phi_{(X,Y)} = H_0 \circ A : H_0(C(X \times Y)) \to H_0(C(X) \otimes C(Y))$ where $A : C(X \times Y) \to C(X) \otimes C(Y)$ is the Alexander-Whitney chain map.

Proof of Eilenberg-Zilber *(continued)*.

- First we address naturality of $\Phi = H_0 \circ A$
- We will actually show that A is natural and since $H_0 : \mathbf{Chain} \to \mathbf{Ab}$ is natural we get naturality of $H_0 \circ A$.
- That is for any morphism $(f, g) : (X, Y) \to (Z, W)$ we get commutative

$$\begin{array}{ccc} C(X \times Y) & \xrightarrow{A} & C(X) \otimes C(Y) \\ \downarrow{\scriptstyle (f\times g)_*} & & \downarrow{\scriptstyle f_* \otimes g_*} \\ C(Z \times W) & \xrightarrow{A} & C(Z) \otimes C(W) \end{array}$$

Proof of Eilenberg-Zilber *(continued).*

- Let $(f, g) : (X, Y) \to (Z, W)$ be a morphism and $\sigma : \Delta^n \to X \times Y$ be a singular simplex.

$$
\begin{aligned}
A(f \times g)_* \sigma &= A(f\pi_X\sigma \times g\pi_Y\sigma) \\
&= \textstyle\sum_{i=0}^n \pi_Z(f\pi_X\sigma \times g\pi_Y\sigma) \circ f_i \otimes \pi_W(f\pi_X\sigma \times g\pi_Y\sigma) \circ b_{n-i} \\
&= \sum_{i=0}^n f\pi_X\sigma \circ f_i \otimes g\pi_Y\sigma \circ b_{n-i}
\end{aligned}
$$

$$
\begin{aligned}
(f_* \otimes g_*)A\sigma &= (f_* \otimes g_*) \sum_{i=0}^n \pi_X\sigma \circ f_i \otimes \pi_Y\sigma \circ b_{n-i} \\
&= \sum_{i=0}^n f\pi_X\sigma \circ f_i \otimes g\pi_Y\sigma \circ b_{n-i}
\end{aligned}
$$

- Thus $A : \mathbf{Top} \times \mathbf{Top} \to \mathbf{Chain}$ is natural.
- Hence $\Phi = H_0 \circ A$ is natural.

Proof of Eilenberg-Zilber *(continued).*

- Next we claim that

$$H_0 \circ A : H_0(C(X \times Y)) \to H_0(C(X) \otimes C(Y))$$

  is an isomorphism.
- For p a point in the space P let $\sigma_p : \Delta^0 \to P$ be the 0-simplex with $\sigma_p(*) = p$.
- Given a singular 0-simplex with $\sigma_{(x,y)}$ of $X \times Y$

$$\begin{aligned} A\sigma_{(x,y)} &= \pi_X \sigma_{(x,y)} f_0 \otimes \pi_Y \sigma_{(x,y)} b_0 = \\ &= \sigma_x f_0 \otimes \sigma_y b_0 \\ &= \sigma_x \otimes \sigma_y \end{aligned}$$

- Let $[\sigma]$ denote the class of σ in H_0
- Then we've shown $H_0 \circ A[\sigma_{(x,y)}] = [A\sigma_{(x,y)}] = [\sigma_x \otimes \sigma_y]$
- $[\sigma_{(x,y)}] = [\sigma_{(x',y')}]$ in $H_0(X \times Y)$
- iff there are paths from x to x' in X and y to y' in Y
- iff $[\sigma_x] = [\sigma_{x'}]$ in $H_0(X)$ and $[\sigma_y] = [\sigma_{y'}]$ in $H_0(Y)$

Proof of Eilenberg-Zilber *(continued).*

- $$\begin{aligned} H_0(S(X,Y)) &= H_0(C(X)\otimes C(Y)) \\ &\cong \bigoplus_i H_i(X)\otimes H_{-i}(Y) \\ &\quad \oplus \bigoplus_i \mathrm{Tor}(H_i(X), H_{-i-1}(Y)) \\ &= H_0(X)\otimes H_0(Y) \end{aligned}$$

- $$\begin{aligned} H_0(T(X,Y)) &= H_0(C(X\times Y)) \\ &= H_0(X\times Y) \end{aligned}$$

- Thus $H_0 \circ A[\sigma_{(x,y)}] = [\sigma_x \otimes \sigma_y] = [\sigma_x] \otimes [\sigma_y]$ is surjective and injective.
- This isomorphism amounts to observation that path components of $X \times Y$ is (setwise) the cartesian product of path components of X with path components of Y.

Proof of Eilenberg-Zilber *(continued).*

- We've fulfilled hypotheses of Acyclic Models Theorem so we get a natural chain map $A' : C(X \times Y) \to C(X) \otimes C(Y)$ defined up to chain homotopy inducing isomorphism $H_0 \circ A$.
- But A is such a natural chain map so $A \simeq A'$.

Proof of Eilenberg-Zilber *(continued).*

- Now we need a futher chain homotopy

$$\gamma : S \to T$$

inducing the reverse isomorphism $H_0(S) \cong H_0(T)$.

- Here we need larger set of models

$$\mathscr{M}_n^S = \left\{ (\Delta^p, \Delta^q) \middle| p + q = n \right\}$$

- Let $e_{p,q}^S \in S_n(\Delta^p, \Delta^q) = \bigoplus_{i \in \mathbf{Z}} C_i(\Delta^p) \otimes C_{n-i}(\Delta^q)$ be

$$e_{p,q}^S = \mathbf{1}_{\Delta^p} \otimes \mathbf{1}_{\Delta^q}$$

- S_n is free with models $\mathscr{M}_n^S$ (check this)

Proof of Eilenberg-Zilber *(continued).*

- Claim that each $(\Delta^p, \Delta^q) \in \mathscr{M}_n^S$ is T-acyclic
- In other words, we must show that for $k > 0$ we have $H_k(T(\Delta^p, \Delta^q)) = 0$
- $$\begin{aligned} H_k(T(\Delta^p, \Delta^q)) &= H_k(C(\Delta^p \times \Delta^q)) \\ &= H_k(\Delta^p \times \Delta^q) \\ &\cong H_k(\{*\}) \\ &= \begin{cases} 0, & k \neq 0 \\ \mathbf{Z}, & k = 0 \end{cases} \end{aligned}$$
- Cross product $- \times -$ is natural and induces isomorphism $H_0(C(X) \otimes C(Y)) \cong H_0(C(X \times Y))$
- Thus it is unique such chain map (up to chain homotopy).

Proof of Eilenberg-Zilber *(continued).*

- Apply acyclic models theorem again noting that both $\mathbf{1}$ and $A \circ (- \times -)$ induce identity on $H_0(C(X) \otimes C(Y))$
- Thus they must agree up to homotopy.
- Finally apply acyclic models theorem once more noting that both $\mathbf{1}$ and $(- \times -) \circ A$ induce identity on $H_0(C(X \times Y))$
- Thus they must agree up to homotopy.
- Thus we may conclude that A and $- \times -$ are chain homotopy inverses.

More applications of Acyclic Model Theorem

- We could have used Acyclic Model Theorem to prove homotopy invariance of H as follows
  - Let T : **Top** $\to$ **Chain** be
    $$T(X) = C(X)$$
  - Let S : **Top** $\to$ **Chain** be
    $$S(X) = C(X \times I)$$
  - $\mathscr{M}_n = \{\Delta^n\}$ with $e_n = \mathbf{1}_{\Delta^n} \in C_n(\Delta^n)$
  - T_n is free with models $\mathscr{M}_n$ (observe definitions)
  - Δ^n is S-acyclic (cone each sing. simplex to fixed $p \in \Delta^n \times I$ to show $H_k(\Delta^n \times I) = 0$ for $k > 0$).
  - Thus there is a unique chain homotopy type of chain maps
    $$\gamma : C(X) \to C(X \times I)$$
    inducing the natural isomorphism $H_0(X) \cong H_0(X \times I)$.

More applications of Acyclic Model Theorem

- Let $h_0, h_1 : X \to X \times I$ be the maps

$$h_0(x) = (x, 0) \qquad h_1(x) = (x, 1)$$

- h_0 and h_1 induce $h_{0\sharp}, h_{1\sharp} : C(X) \to C(X \times I)$ which in turn induce the natural isomorphism $H_0(X) \cong H_1(X \times I)$
- Hence $h_{0\sharp}$ and $h_{1\sharp}$ are chain homotopic.
- Now suppose

$$f_0, f_1 : X \to Y$$

  are homotopic
- Have homotopy $F : X \times I \to Y$ from f_0 to f_1 with $f_0 = F \circ h_0$ and $f_1 = F \circ h_1$
- Hence

$$f_{0\sharp} = F_\sharp \circ h_{0\sharp} \simeq F_\sharp \circ h_{1\sharp} = f_{1\sharp}$$