Algebraic Topology II – Lecture 13

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1 Proof of Acyclic Models Theorem

Definition 1.1 (Free functor with respect to the models)**.** Given a category $\mathcal{C}$ with models $\mathcal{M}$ a functor

$$T : \mathcal{C} \to \mathbf{Ab}$$

is **free with respect to** $\mathcal{M}$ if for each $M \in \mathcal{M}$ there is $e_M \in T(M)$ such that for all $X \in \mathrm{Ob}(\mathcal{C})$

$$\left\{ T(f)(e_M) \middle| M \in \mathcal{M}, f \in \mathrm{Mor}(M, X) \right\}$$

is a free basis for $T(X)$.

Definition 1.2 (Free functor with respect to the models)**.** Given a category $\mathcal{C}$ with models $\mathcal{M} = \cup_n \mathcal{M}_n$ a functor

$$T : \mathcal{C} \to \mathbf{Chain}$$

is **free with respect to** $\mathcal{M} = \cup_n \mathcal{M}_n$ if each T_n is free with respect to $\mathcal{M}_n$

Theorem 1.3 (Acyclic models theorem)**.**

- *$\mathcal{C}$ a category with models $\mathcal{M} \subset \mathcal{C}$*
- *$S, T : \mathcal{C} \to$ **Chain** functors with $S_n = T_n = 0$ for $n < 0$*
- *T is free with respect to the models $\mathcal{M} = \cup_n \mathcal{M}_n$*
- *$H_i(S(M)) = 0$ for $i > 0$ for all models $M \in \mathcal{M}$ (M is S-**acyclic**)*
- *There is a natural transformation*
$$\Phi : H_0(T) \to H_0(S)$$

Then there is a natural transformation

$$\varphi : T \to S$$

which induces Φ unique up to chain homotopy.

Proof.

- Let $S, T : \mathcal{C} \to$ **Chain** satisfy the assumptions of the theorem.
- We want a natural transformation $\varphi : T \to S$ inducing

$$\Phi : H_0(T) \to H_0(S)$$

- That is, for each $X \in \mathcal{C}$ we want a chain map
$$\varphi_X : T(X) \to S(X)$$
such that for any $f : X \to Y$ we get commutative
$$\begin{array}{ccc} T(X) & \xrightarrow{T(f)} & T(Y) \\ {\scriptstyle \varphi_X}\downarrow & & \downarrow{\scriptstyle \varphi_Y} \\ S(X) & \xrightarrow{S(f)} & S(Y) \end{array}$$
Furthermore, φ must induce the given natural transformation
$$\Phi : H_0(T) \to H_0(S)$$

- In constructing the natural transformation $\varphi^n : T_n \to S_n$ we will make some arbitrary choices (one for each model $M_n \in \mathcal{M}_n$).
- After making these choices naturality will impose a definition for $\varphi^n_X : T_n(X) \to S_n(X)$ for all $X \in \mathcal{C}$.
- First, the arbitrary choice for each model $M_0 \in \mathcal{M}_0$:
  - Let $M_0 \in \mathcal{M}_0$ be a model of $\mathcal{C}$
  - T_0 is free with models $\mathcal{M}_0$ so we have distinguished element $e_{M_0} \in T_0(M_0)$.
  - $T_{-1}(M_0) = 0$ so $\partial e_{M_0} = 0$. Let $[e_{M_0}]$ denote its class in $H_0(M_0)$.

$$\begin{array}{ccc} {\scriptstyle e_{M_0}} & & {\scriptstyle [e_{M_0}]} \\ T_0(M_0) & \longrightarrow & H_0(T(M_0)) \\ & & {\scriptstyle \Phi_{M_0}}\downarrow{\scriptstyle \cong} \\ S_0(M_0) & \longrightarrow & H_0(S(M_0)) \\ {\scriptstyle s_{M_0}} & & {\scriptstyle \Phi([e_{M_0}])} \end{array}$$

  - Choose $s_{M_0} \in S_0(M_0)$ so that $[s_{M_0}] = \Phi_{M_0}([e_{M_0}])$. In general there is lots of flexibility in our choice of s_{M_0}.
- Now let $X \in \mathcal{C}$ be any object.
- Recall that
$$\{T_0(f)(e_{M_0}) | M_0 \in \mathcal{M}_0 \text{ and } f \in \mathrm{Mor}(M_0, X)\}$$
is a free basis for $T_0(X)$.
- We define $\varphi^0_X : T_0(X) \to S_0(X)$ to be the unique homomorphism satisfying
$$\varphi^0_X(T_0(f)(e_{M_0})) = S_0(f)(s_{M_0}).$$
- Suppose $\gamma : X \to Y$ is a morphism.
- Then for each basis element $T_0(f)(e_{M_0}) \in T_0(X)$ we have
$$\begin{aligned} \varphi^0_Y T_0(\gamma) T_0(f)(e_{M_0}) &= \varphi^0_Y T_0(\gamma \circ f)(e_{M_0}) \\ &= S_0(\gamma \circ f)(s_{M_0}) \\ &= S_0(\gamma) S_0(f)(s_{M_0}) \\ &= S_0(\gamma) \varphi^0_X T_0(f)(e_{M_0}) \end{aligned}$$

- Hence $\varphi^0_Y T_0(\gamma) = S_0(\gamma)\varphi^0_X$ so $\varphi^0 : T_0 \to S_0$ is natural.
- We claim that φ^0_X induces $\Phi_X : H_0(T(X)) \to H_0(S(X))$.
  - Recall that Φ is natural so given $f : M_0 \to X$ we have
$$\Phi_X H_0T(f) = H_0S(f)\Phi_{M_0}$$
  - Furthermore $\varphi^0 : T_0 \to S_0$ is natural so
$$\varphi^0_X T_0(f) = S_0(f)\varphi^0_{M_0}$$
  - Lastly, recall we chose s_{M_0} so that $\Phi_{M_0}[e_{M_0}] = [s_{M_0}]$
  - Thus for each basis element $T_0(f)(e_{M_0}) \in T_0(X)$ we have
$$\begin{aligned}[\varphi^0_X T_0(f)(e_{M_0})] &= [S_0(f)\varphi^0_{M_0}(e_{M_0})] \\ &= [S_0(f)(s_{M_0})] \\ &= H_0S(f)[s_{M_0}] \\ &= H_0S(f)\Phi_{M_0}[e_{M_0}] \\ &= \Phi_X H_0T(f)[e_{M_0}] \\ &= \Phi_X[T_0(f)(e_{M_0})]\end{aligned}$$
  - Hence φ^0_X induces Φ_X.
- Next, the arbitrary choice for each model $M_1 \in \mathcal{M}_1$:
  - Let $M_1 \in \mathcal{M}_1$ be a model of $\mathcal{C}$
  - T_1 is free with models $\mathcal{M}_1$ so we have distinguished element $e_{M_1} \in T_1(M_1)$.
  - We φ^0 induces Φ so we have commutative:
$$\begin{array}{ccccccc} \overset{e_{M_1}}{T_1(M_1)} & \xrightarrow{\partial} & \overset{\partial e_{M_1}}{T_0(M_1)} & \xrightarrow{[\cdot]} & \overset{0}{H_0(T(M_1))} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi^0_{M_1}} & & \downarrow{\scriptstyle \Phi_{M_1}\ \cong} & & \\ \underset{s_{M_1}}{S_1(M_1)} & \xrightarrow{\partial} & \underset{\varphi^0_{M_1}\partial e_{M_1}}{S_0(M_1)} & \xrightarrow{[\cdot]} & \underset{0}{H_0(S(M_1))} & \longrightarrow & 0 \end{array}$$
    with exact rows.
  - $[\varphi^0_{M_1}\partial e_{M_1}] = \Phi_{M_1}[\partial e_{M_1}] = \Phi_{M_1}0 = 0$.
  - By exactness of bottom row there is $s_{M_1} \in S_1(M_1)$ so that $\partial s_{M_1} = \varphi^0_{M_1}\partial e_{M_1}$. In general there is lots of flexibility in our choice of s_{M_1} for model M_1.
- Now let $X \in \mathcal{C}$ be any object.
- We define $\varphi^1_X : T_1(X) \to S_1(X)$ on the basis $\{T_1(f)(e_{M_1})\}$ of $T_1(X)$ to be the unique homomorphism satisfying
$$\varphi^1_X(T_1(f)(e_{M_1})) = S_1(f)(s_{M_1}).$$
- Again, given a morphism $\gamma : X \to Y$, we have for each basis element $T_1(f)(e_{M_1}) \in T_1(X)$ we have
$$\varphi^1_Y T_n(\gamma)T_1(f)(e_{M_1}) = S_1(\gamma)\varphi^1_X T_n(f)(e_{M_1}).$$

  Hence $\varphi^1 : T_1 \to S_1$ is a natural transformation.

- We claim that for any $X \in \mathcal{C}$ we have $\partial\varphi^1_X = \varphi^0_X\partial$.
  - Given $f : M_1 \to X$ we get chain maps $T(f) : T(M_1) \to T(X)$ and $S(f) : S(M_1) \to S(X)$.
  - Furthermore $\varphi^0 : T_0 \to S_0$ is natural so

$$\varphi^0_X T_0(f) = S_0(f)\varphi^0_{M_1}$$

  - Thus for each basis element $T_1(f)(e_{M_1}) \in T_1(X)$ we have

$$\begin{aligned}
\partial\varphi^1_X T_1(f)(e_{M_1}) &= \partial S_1(f)(s_{M_1}) \\
&= S_0(f)\partial(s_{M_1}) \\
&= S_0(f)\varphi^0_{M_1}\partial(e_{M_1}) \\
&= \varphi^0_X T_0(f)\partial(e_{M_1}) \\
&= \varphi^0_X \partial T_1(f)(e_{M_1})
\end{aligned}$$

  - Hence $\partial\varphi^1_X = \varphi^0_X\partial$.
  - For $k < 0$ define $\varphi^k : T_k \to S_k$ to be the 0-map.
  - Observe for the purposes of the induction below that $T_{-1}(X) = S_{-1}(X) = 0$ so $\partial\varphi^0_X = 0 = \varphi^{-1}_X\partial$.
- Now let $n \geq 2$ and assume for each $k < n$ we have natural transformations $\varphi^k : T_k \to S_k$ such that $\partial\varphi^k = \varphi^{k-1}\partial$.
- First we make our arbitrary choice for each model $M_n \in \mathcal{M}_n$:
  - Let $M_n \in \mathcal{M}_n$ be a model of $\mathcal{C}$ and consider $e_{M_n} \in T_n(M_n)$.

$$\begin{array}{ccccc}
\overset{e_{M_n}}{T_n(M_n)} & \xrightarrow{\partial} & \overset{\partial e_{M_n}}{T_{n-1}(M_n)} & \xrightarrow{\partial} & \overset{0}{T_{n-2}(M_n))} \\
 & & \Big\downarrow {\scriptstyle \varphi^{n-1}_{M_n}} & & \Big\downarrow {\scriptstyle \varphi^{n-2}_{M_n}} \\
\underset{s_{M_n}}{S_n(M_n)} & \xrightarrow{\partial} & \underset{\varphi^{n-1}_{M_n}\partial e_{M_n}}{S_{n-1}(M_n)} & \xrightarrow{\partial} & \underset{0}{S_{n-2}(M_n)}
\end{array}$$

  - $\varphi^{n-1}_{M_n}\partial e_{M_n} \in S_{n-1}(M_n)$ is a cycle since

$$\partial\varphi^{n-1}_{M_n}\partial e_{M_n} = \varphi^{n-2}_{M_n}\partial\partial e_{M_n} = 0.$$

  - By S-acyclicity of M_n we have $H_{n-1}(S(M_n)) = 0$ so cycles in $S_{n-1}(M_n)$ must be boundaries.
  - Hence there is $s_{M_n} \in S_n(M_n)$ with $\partial s_{M_n} = \varphi^{n-1}_{M_n}\partial e_{M_n}$. Again in general there is lots of freedom in our choice of s_{M_n} for the model M_n.
- Now let $X \in \mathcal{C}$ be any object.
- We define $\varphi^n_X : T_n(X) \to S_n(X)$ on the basis $\{T_n(f)(e_{M_n})\}$ of $T_n(X)$ to be the unique homomorphism satisfying

$$\varphi^n_X(T_n(f)(e_{M_n})) = S_n(f)(s_{M_n}).$$

- As with the case $n = 0$ above, given a morphism $\gamma : X \to Y$, for each basis element $T_n(f)(e_{M_n}) \in T_n(X)$ we have

$$\varphi^n_Y T_n(\gamma)T_n(f)(e_{M_n}) = S_n(\gamma)\varphi^n_X T_n(f)(e_{M_n}).$$

  Hence $\varphi^n : T_n \to S_n$ is a natural transformation.

- Furthermore as with the case $n = 1$ above we have $\partial\varphi_X^n T_n(f)(e_{M_n}) = \varphi_X^{n-1}\partial T_n(f)(e_{M_n})$ so the chain map condition holds at level n and below.
- By induction we have established the existence of the natural transformation $\varphi : T \to S$ inducing $\Phi : H_0T \to H_0S$.
- Suppose $\varphi' : T \to S$ is another natural transformation inducing Φ.
- We will give a natural construction for each $X \in \mathcal{C}$ of a chain homotopy $P_X : T(X) \to S(X)$ satisfying

$$\partial P_X + P_X\partial = \varphi_X - \varphi'_X$$

  - Suppose $P_k : T_k \to S_{k+1}$ is defined for all $X \in \mathcal{C}$ and $k < n$
  - Claim that
$$\varphi_{M_n}(e_{M_n}) - \varphi'_{M_n}(e_{M_n}) - P_{M_n}(\partial e_{M_n})$$
    is a cycle in $S_n(M_n)$
  -
$$\begin{aligned}\partial(\varphi_{M_n}(e_{M_n}) - \varphi'_{M_n}(e_{M_n}) - P_{M_n}(\partial e_{M_n}))\\ &= \partial\varphi_{M_n}(e_{M_n}) - \partial\varphi'_{M_n}(e_{M_n}) - (\partial P_{M_n})(\partial e_{M_n})\\ &= \varphi_{M_n}\partial(e_{M_n}) - \varphi'_{M_n}\partial(e_{M_n}) - (\varphi_{M_n} - \varphi'_{M_n} - P_{M_n}\partial)(\partial e_{M_n})\\ &= 0\end{aligned}$$
  - M_n is S-acyclic so there is $s_{M_n}^{n+1} \in S_{n+1}(M_n)$ with
$$\partial s_{M_n}^{n+1} = \varphi_{M_n}(e_{M_n}) - \varphi'_{M_n}(e_{M_n}) - P_{M_n}$$
  - For a space X we have free basis for $T_n(X)$
$$\left\{T_n(f)(e_{M_n})\middle| M_n \in \mathcal{M}_n \text{ and } f \in \operatorname{Mor}(M_n, X)\right\}$$
  - Set
$$P_n(T_n(f)(e_{M_n})) = S(f)(s_{M_n}^{n+1})$$
  - If $n = 0$ use fact that φ and φ' induce Φ to show $\varphi_{M_0}(e_{M_0}) - \varphi_{M_0}(e_{M_0})$ is a cycle in $S_0(M_0)$.
  - Similar argument to one above showing φ is natural shows P is natural.
  - Similar argument to one above showing φ is a chain map shows P is a chain homotopy.

$\square$