Algebraic Topology II – Lecture 14

Nathan Broaddus

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1 More applications of the Acyclic Models Theorem

Recall homotopy invariance of homology groups:

Theorem 1.1 (Homotopy invariance of homology). *If $f_0, f_1 : X \to Y$ are homotopy equivalent maps then the induced maps $f_{0*}, f_{1*} : H_*(X) \to H_*(Y)$ agree.*

An alternate proof sketch using the Acyclic Models Theorem is as follows:

Proof.

- Let $T : \mathbf{Top} \to \mathbf{Chain}$ be
$$T(X) = C(X)$$
- Let $S : \mathbf{Top} \to \mathbf{Chain}$ be
$$S(X) = C(X \times I)$$
- $\mathcal{M}_n = \{\Delta^n\}$ with $e_n = \mathrm{id}_{\Delta^n} \in C_n(\Delta^n)$

- T_n is free with models $\mathcal{M}_n$ (observe definitions)
- Δ^n is S-acyclic (cone each sing. simplex to fixed $p \in \Delta^n \times I$ to show $H_k(\Delta^n \times I) = 0$ for $k > 0$).
- Thus there is a unique chain homotopy type of chain maps
$$\gamma : C(X) \to C(X \times I)$$
inducing the natural isomorphism $H_0(X) \cong H_0(X \times I)$.
- Let $i_0, i_1 : X \to X \times I$ be the inclusion maps
$$i_0(x) = (x, 0) \qquad i_1(x) = (x, 1)$$
- i_0 and i_1 induce $i_{0\sharp}, i_{1\sharp} : C(X) \to C(X \times I)$ which in turn induce the natural isomorphism $H_0(X) \cong H_0(X \times I)$
- Hence $i_{0\sharp}$ and $i_{1\sharp}$ are chain homotopic.
- Now suppose
$$f_0, f_1 : X \to Y$$
are homotopic
- Have homotopy $F : X \times I \to Y$ from f_0 to f_1 with $f_0 = F \circ i_0$ and $f_1 = F \circ i_1$

- $i_{0\sharp}$ and $i_{1\sharp}$ are chain homotopic.
- So composing with the chain map $F_\sharp$ we get a chain homotopy:

$$F_\sharp \circ i_{0\sharp} \simeq F_\sharp \circ i_{1\sharp}$$

- Hence

$$f_{0\sharp} = F_\sharp \circ i_{0\sharp} \simeq F_\sharp \circ i_{1\sharp} = f_{1\sharp}$$

- Notice we don't know what chain homotopy between $i_{0\sharp}$ and $i_{1\sharp}$ looks like.

□

Definition 1.2 (Singular cubical homology)**.** Let X be a space. The set of **singular** n-cubes of X is

$$\mathcal{Q}_n(X) = \{\gamma | \gamma : I^n \to X \text{ continuous}\}.$$

Let

$$Q_n(X) = \mathbf{Z}[\mathcal{Q}_n(X)]$$

be the free abelian group on singulars n-cubes of X. Let $a_i, b_i : I^{n-1} \to I^n$ be the functions

$$a_i(x_1, \cdots, x_{n-1}) = (x_1, \cdots, x_{i-1}, 0, x_i, \cdots, x_{n-1})$$
$$b_i(x_1, \cdots, x_{n-1}) = (x_1, \cdots, x_{i-1}, 1, x_i, \cdots, x_{n-1})$$

For $\gamma : I^n \to X$ a singular n-simplex we define $\partial\gamma \in Q_{n-1}(X)$ to be

$$\partial\gamma = \sum_{i=1}^{n} (-1)^i (\gamma \circ a_i - \gamma \circ b_i).$$

Theorem 1.3 (Singular cubical homology and singular simplicial homology agree)**.** *Let X be a space. We have a natural isomorphism*

$$H_n(X) \cong H_n(Q(X)).$$

Proof.

- Singular simplicial chains C have models $\mathcal{M}_n = \{\Delta_n\}$
- One may prove that these models are Q-acyclic by coning to a point.
- This gives chain map $c : C \to Q$
- Singular cubical chains C have models $\mathcal{M}_n = \{I^n\}$
- These models are contractible and hence C-acyclic
- This gives chain map $q : Q \to C$.
- $H_0(X) = H_0(Q(X))$ since $Q_0(X) = C_0(X)$ and $Q_1(X) = C_1(X)$.
- Both 1 and $c \circ q$ induce the identity map on $H_0(Q(X))$ so $c \circ q \simeq 1$.
- Both 1 and $q \circ c$ induce the identity map on $H_0(X)$ so $q \circ c \simeq 1$.

□

2 Products in homology and cohomology

2.1 The cross product on homology

Having established the Eilenberg-Zilber Theorem (see Lecture 11) we can go back and forth between $C(X) \otimes C(Y)$ and $C(X \times Y)$ at will.

Definition 2.1 (Cross product on homology)**.** If X and Y are spaces then we have the **cross product** $- \times -$ on their homology groups

$$- \times - : H_p(X) \otimes H_q(Y) \to H_{p+q}(X \times Y)$$

which is induced by the cross product

$$- \times - : C_p(X) \otimes C_q(Y) \to C_{p+q}(X \times Y)$$

we defined at the singular simplicial chain level (See Lecture 11). Note we may also view the cross product as a **bilinear** map

$$- \times - : H_p(X) \times H_q(Y) \to H_{p+q}(X \times Y)$$

Note: if σ is a p-simplex and η is a q-simplex then $\sigma \times \eta$ has $\binom{p+q}{p}$ terms.

2.2 The cross product on cohomology

- Let X and Y be spaces and G_1 and G_2 be abelian groups
- Given cochains
$$u \in C^p(X; G_1) = \mathrm{Hom}(C_p(X), G_1)$$
and
$$v \in C^q(Y; G_2) = \mathrm{Hom}(C_q(Y), G_2)$$
- we get the homomorphism
$$u \otimes v \in \mathrm{Hom}(C_p(X) \otimes C_q(Y), G_1 \otimes G_2)$$
- Understanding $u \otimes v$ to be 0 on $C_i(X) \otimes C_j(Y)$ if $i \neq p$ or $j \neq q$ we have
$$u \otimes v \in \mathrm{Hom}((C(X) \otimes C(Y))_{p+q}, G_1 \otimes G_2)$$
- We have the Alexander-Whitney map (See Lecture 11)
$$A : C(X \times Y) \to C(X) \otimes C(Y)$$
- Dualizing with $G_1 \otimes G_2$ we have the dual of the Alexander-Whitney map:
$$A^* : (C(X) \otimes C(Y))^* \to C^*(X \times Y)$$

Definition 2.2 (Cross product on cochains)**.** For cochains

$$u \in C^p(X; G_1) = \mathrm{Hom}(C_p(X), G_1)$$

$$v \in C^q(Y; G_2) = \mathrm{Hom}(C_q(Y), G_2)$$

the **cross product** $u \times v \in C^{p+q}(X \times Y; G_1 \otimes G_2)$ is

$$u \times v = A^*(u \otimes v)$$

Where A^* is the dual of the Alexander-Whitney map dualized by $G_1 \otimes G_2$.

- The Alexander-Whitney map A is a chain map so its dual A^* is a cochain map.

$$A^* : (C(X) \otimes C(Y))^* \to C^*(X \times Y)$$

- Note that the coboundary δ for $(C(X) \otimes C(Y))^*$ is the dual $\delta = \partial^*$ where ∂ is the boundary for $C(X) \otimes C(Y)$.
- Recall if $a \in C_p(X)$ and $b \in C_q(Y)$ we have

$$\partial(a \otimes b) = \partial a \otimes b + (-1)^p a \otimes \partial b$$

- We show in Lemma 2.3 below that if $u \in C^p(X; G_1)$ and $v \in C^q(X; G_2)$ we get

$$\delta(u \otimes v) = (\delta u) \otimes v + (-1)^p u \otimes \delta v$$

- Applying the dual of the Alexander-Whitney map we arrive at

$$\delta(u \times v) = (\delta u) \times v + (-1)^p u \times \delta v$$

Lemma 2.3. *In the cochain complex $(C(X) \otimes C(Y))^*$ if $u \in C^p(X)$ and $v \in C^q(Y)$ then*

$$\delta(u \otimes v) = (\delta u) \otimes v + (-1)^p u \otimes \delta v$$

Proof.
- Suppose $u \in C^p(X)$ and $v \in C^q(Y)$.
- For $a \in C_*(X)$ and $b \in C_*(Y)$ we have

$$\begin{aligned}(\delta(u \otimes v))(a \otimes b) &= (u \otimes v)\partial(a \otimes b)\\ &= (u \otimes v)(\partial a \otimes b + (-1)^{|a|} a \otimes \partial b)\\ &= u\partial a \otimes vb + (-1)^{|a|} ua \otimes v\partial b\end{aligned}$$

$$\begin{aligned}&((\delta u) \otimes v + (-1)^p u \otimes \delta v)(a \otimes b)\\ &= (\delta u)a \otimes vb + (-1)^p ua \otimes \delta vb\\ &= u\partial a \otimes vb + (-1)^p ua \otimes v\partial b\end{aligned}$$

- Note that $ua = 0$ unless $|a| = p$.

□

Lemma 2.4.

1. *If u and v are cocycles then $u \times v$ is a cocycle.*
2. *Given cohomologous cocycles u_1 and u_2 and cocycle v we get $u_1 \times v$ cohomologous to $u_2 \times v$.*
3. *Given cohomologous cocycles v_1 and v_2 and cocycle u we get $u \times v_1$ cohomologous to $u \times v_2$.*

This lemma is an immediate consequence of our formula for $\delta(u \times v)$ and implies that the cross product on cochains induces a cross product on cohomology.

Definition 2.5 (Cross product on cohomology)**.** The cross product on cochains

$$- \times - : C^p(X; G_1) \otimes C^q(Y; G_2) \to C^{p+q}(X \times Y, G_1 \otimes G_2)$$

induces the **cross product on cohomology**

$$- \times - : H^p(X; G_1) \otimes H^q(Y; G_2) \to H^{p+q}(X \times Y, G_1 \otimes G_2)$$

- For $[u] \in H^p(X)$ and $[v] \in H^q(Y)$ represent $[u]$ and $[v]$ as cocycles

$$u : C_p(X) \to G_1 \quad \text{and} \quad v : C_q(Y) \to G_2.$$

- If $a \in C_p(X)$ and $b \in C_q(Y)$ then $u \otimes v : C_p(X) \otimes C_q(Y) \to G_1 \otimes G_2$ satisfies

$$(u \otimes v)(a \otimes b) = u(a) \otimes v(b)$$

- $u \times v = A^*(u \otimes v)$ so given $\sigma : \Delta^{p+q} \to X \times Y$

$$\begin{aligned}
(u \times v)(\sigma) &= A^*(u \otimes v)(\sigma) \\
&= (u \otimes v)A(\sigma) \\
&= (u \otimes v)\sum_{i=0}^{p+q}(\pi_X \sigma f_i) \otimes (\pi_Y \sigma b_{p+q-i}) \\
&= \sum_{i=0}^{p+q}(u \otimes v)(\pi_X \sigma f_i) \otimes (\pi_Y \sigma b_{p+q-i}) \\
&= \sum_{i=0}^{p+q} u(\pi_X \sigma f_i) \otimes v(\pi_Y \sigma b_{p+q-i}) \\
&= u(\pi_X \sigma f_p) \otimes v(\pi_Y \sigma b_q)
\end{aligned}$$

We have proven the following:

Lemma 2.6. *For $u \in C^p(X; G_1)$, $v \in C^q(Y; G_2)$ and $\sigma \in C^{p+q}(X \times Y)$*

$$(u \times v)(\sigma) = u(\pi_X \sigma f_p) \otimes v(\pi_Y \sigma b_q)$$

Proposition 2.7 (Properties of the cohomological cross product)**.** *Given $u \in H^p(X; G_1)$, $v \in H^q(Y; G_2)$ and $w \in H^r(Z; G_3)$ we have:*

1. *Associativity*

$$u \times (v \times w) = (u \times v) \times w.$$

2. *Commutativity*

$$s^*(u \times v) = (-1)^{pq} v \times u.$$

where $s : Y \times X \to X \times Y$ is the continuous map swapping factors.

3. *Existence of units*

$$\begin{aligned}
u \times 1_Y &= \pi_Y^*(u) \\
1_X \times v &= \pi_X^*(v)
\end{aligned}$$

where $1_W \in H^0(W; \mathbf{Z})$ is the cohomology class of the augmentation map $\varepsilon : C_0(W) \to \mathbf{Z}$ and $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the projections.

2.3 The cup product in cohomology

Definition 2.8 (Pairing of abelian groups)**.** A **pairing** of the abelian groups G_1 and G_2 is a homomorphism

$$\langle , \rangle : G_1 \otimes G_2 \to G_3$$

or equivalently a bilinear map $\langle , \rangle : G_1 \times G_2 \to G_3$.

Example 2.9. For any ring R we have the pairing $\langle , \rangle : R \times R \to R$ given by

$$\langle r_1, r_2 \rangle = r_1 r_2.$$

For any space X we have the **diagonal map**:

$$\Delta : X \to X \times X$$

where $\Delta(x) = (x, x)$

Definition 2.10 (Cup product)**.** Let X be a space and fix a pairing $\omega : G_1 \otimes G_2 \to G_3$. Given

$$u \in H^p(X; G_1) \text{ and } v \in H^q(X; G_2)$$

their **cup product** $u \smile v \in H^{p+q}(X; G_3)$ is given by

$$u \smile v = \omega \Delta^*(u \times v)$$

where $\Delta^* : X \to X \times X$ is the diagonal map.

If $G_1 = G_2 = R$ is a ring the pairing ω is understood to be multiplication in R unless otherwise noted.