Algebraic Topology II – Lecture 15

Nathan Broaddus

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1 Cup product on cohomology

1.1 The cup product in cohomology

Definition 1.1 (Pairing of abelian groups)**.** A **pairing** of the abelian groups G_1 and G_2 is a homomorphism

$$\langle , \rangle : G_1 \otimes G_2 \to G_3$$

or equivalently a bilinear map $\langle , \rangle : G_1 \times G_2 \to G_3$.

Example 1.2 (Universal pairing). For any abelian groups G_1 and G_2 we have the universal pairing

$$\omega : G_1 \otimes G_2 \to G_1 \otimes G_2.$$

where $\omega = \mathrm{id}_{G_1 \otimes G_2}$.

Example 1.3 (Pairing for a ring). For any (possibly noncommutative) ring R we have the paring $\langle , \rangle : R \times R \to R$ given by

$$\langle r_1, r_2 \rangle = r_1 r_2.$$

Definition 1.4 (Cup product)**.** Let X be a space and fix a pairing $\omega : G_1 \otimes G_2 \to G_3$. Given

$$u \in H^p(X; G_1) \text{ and } v \in H^q(X; G_2)$$

their **cup product** $u \smile v \in H^{p+q}(X; G_3)$ is given by

$$u \smile v = \omega \Delta^*(u \times v)$$

where $\Delta : X \to X \times X$ is the diagonal map. If $G_1 = G_2 = R$ is a ring the pairing ω is understood to be multiplication in R unless otherwise noted.

Proposition 1.5 (Properties of the cohomological cup product)**.** *Given a space X and a commutative ring with unit R and for $u \in H^p(X; R)$, $v \in H^q(X; R)$ and $w \in H^r(X; R)$ we have:*

1. *Associativity*
$$u \smile (v \smile w) = (u \smile v) \smile w.$$

2. *Commutativity*
$$u \smile v = (-1)^{pq} v \smile u.$$

3. *Existence of unit*
$$u \smile 1_X = u$$
where $1_X \in H^0(X; R)$ is the cohomology class of the augmentation map $\varepsilon : C_0(X) \to R$ sending all singular simplices to the unit in R.

Corollary 1.6 (Cohomology ring)**.** *For any space X and commutative ring with unit R the cohomology $H^*(X;R)$ is a commutative graded algebra over R with multiplication given by the cup product*

$$- \smile - : H^p(X;R) \times H^q(X;R) \to H^r(X;R)$$

and unit $1_X = [\varepsilon] \in H^0(X;R)$ where $\varepsilon : C_0(X) \to R$ is the augmentation map. This ring is called the **cohomology ring** *of X.*

Proof.

- $H^*(X;R)$ is an R-module since $C^*(X;R)$ is.
- Multiplicative ring axioms follow from properties of the cohomological cup product above.
- Distributive ring axiom follows from bilinearity of cup product.

□

1.2 The cohomology in practice

Lemma 1.7. *For any space X and commutative ring with unit R for $u \in C^p(X;R)$, $v \in C^q(X;R)$ and $\sigma \in C_{p+q}(X)$*

$$u \smile v(\sigma) = u(\sigma \circ f_p) \cdot v(\sigma \circ b_q)$$

Proof.

$$\begin{aligned} u \smile u(\sigma) &= \omega\Big(\Delta^*(u \times v)(\sigma)\Big) \\ &= \omega\Big((u \times v)\Delta(\sigma)\Big) \\ &= \omega\Big((u\pi_1\Delta\sigma f_p) \otimes (v\pi_2\Delta\sigma b_q)\Big) \\ &= \omega\Big(u(\sigma f_p) \otimes v(\sigma b_q)\Big) \\ &= u(\sigma \circ f_p) \cdot v(\sigma \circ b_q) \end{aligned}$$

□

- Universal coefficient theorem gives cohomology in terms of homology as an abelian group.
- Full description of $H^*(X;R)$ should include R-algebra structure and generators.
- We now examine some cohomology computations of cohomology rings.
- For any space X
$$C^n(X) \cong \{f \mid f : \mathcal{S}_n(X) \to \mathbf{Z} \text{ a function}\}$$
where $\mathcal{S}_n(X)$ is set of singular n-simplices of X.
- 0-simplices of X are 1-1 with points of X so
$$C^0(X) \cong \{f \mid f : X \to \mathbf{Z} \text{ a function}\}$$
- Let $f \in C^0(X)$ and $\sigma : \Delta^1 \to X$ be a singular simplex.
- Then
$$\delta f(\sigma) = f(\partial\sigma) = f(\sigma(1)) - f(\sigma(0))$$
- Thus $\delta f = 0$ precisely when $f(x_1) = f(x_2)$ whenever x_1 and x_2 in same path component of X

- So $H^0(X) \subset C^0(X)$ is the subset

$$H^0(X) \cong \{f \mid f : X \to \mathbf{Z} \text{ constant on path components of } X\}$$

- Let $A, B \subset X$ be unions of path components of X. Set

$$\varepsilon_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

- Given $x \in X$

$$\begin{aligned} \varepsilon_A \smile \varepsilon_B(x) &= \varepsilon_A(x) \cdot \varepsilon_B(x) \\ &= \begin{cases} 1, & x \in A \cap B \\ 0, & x \notin A \cap B \end{cases} \\ &= \varepsilon_{A \cap B}(x) \end{aligned}$$

  Hence $\varepsilon_A \smile \varepsilon_A = \varepsilon_A$ is an **idempotent** of $H^*(X)$.

Proposition 1.8. *For each path component A of X*

$$H^0(A) \cong \langle \varepsilon_A \mid \varepsilon_A^2 = \varepsilon_A \rangle \cong \mathbf{Z}$$

$H^0(X)$ has presentation:

$$H^0(X) \cong \prod_{A \text{ path comp. of } X} \langle \varepsilon_A \rangle \Big/ \langle \varepsilon_A^2 = \varepsilon_A, \varepsilon_A \varepsilon_B = 0 \text{ for } A \neq B \rangle$$

- Now suppose $u \in C^p(X)$ and A is a path component of X.
- Then given $\sigma \in \mathcal{S}_p(X)$

$$\begin{aligned} \varepsilon_A \smile u(\sigma) &= \varepsilon_A(\sigma \circ f_0) \cdot u(\sigma \circ b_p) \\ &= \begin{cases} u(\sigma), & \text{Image}\, \sigma \subset A \\ 0, & \text{Image}\, \sigma \not\subset A \end{cases} \end{aligned}$$

  Hence $\varepsilon_A \smile u = u \smile \varepsilon_A = u|_{C_p(A)}$.

Proposition 1.9. *Let A be a path component of X. Then the inclusion $i : A \to X$ induces the surjective ring homomorphism*

$$i^* : H^*(X) \to H^*(A)$$

and for any $u \in H^*(X)$

$$i^*(u) = \varepsilon_A \smile u.$$

Example 1.10 (Cohomology ring of S^1).

- Let $p : \mathbf{R} \to S^1$ be the covering map $p(t) = e^{2\pi i t}$
- For each $n \in \mathbf{Z}$ and singular 1-simplex $\sigma : I \to S^1$ let $\tilde{\sigma} : I \to \mathbf{R}$ be a lift of σ and set $s_{\tilde{\sigma}} = \tilde{\sigma}(0)$ and $e_{\tilde{\sigma}} = \tilde{\sigma}(1)$
- Set $u : \mathcal{S}_1(S^1) \to \mathbf{Z}$ to be

$$u(\sigma) = \begin{cases} |\mathbf{Z} \cap (s_{\tilde{\sigma}}, e_{\tilde{\sigma}}]|, & s_{\tilde{\sigma}} \leq e_{\tilde{\sigma}} \\ -|\mathbf{Z} \cap [e_{\tilde{\sigma}}, s_{\tilde{\sigma}})|, & s_{\tilde{\sigma}} > e_{\tilde{\sigma}} \end{cases}$$

- If $\sigma : \Delta^2 \to S^1$ is a singular 2-simplex we have a lift $\tilde{\sigma} : \Delta^2 \to \mathbf{R}$.

$$\begin{aligned}
\delta u \sigma &= u(\partial \sigma) \\
&= u(\sigma|_{[x_1,x_2]} - \sigma|_{[x_0,x_2]} + \sigma|_{[x_0,x_1]}) \\
&= u(\tilde{\sigma}|_{[x_1,x_2]}) - u(\tilde{\sigma}|_{[x_0,x_2]}) + u(\tilde{\sigma}|_{[x_0,x_1]}) \\
&= |\mathbf{Z} \cap (a_1, a_2]| - |\mathbf{Z} \cap (a_0, a_2]| + |\mathbf{Z} \cap (a_0, a_1]| \\
&= 0
\end{aligned}$$

- $H^*(S^1) \cong \mathbf{Z}[u] \big/ \langle u^2 = 0 \rangle$ where $|u| = 1$

Example 1.11 (Cohomology ring of the n-torus).

- Claim:

$$H^*(T^n) \cong \mathbf{Z}[u_1, \cdots, u_n] \big/ \langle u_i^2 = 0 \rangle \cong \Lambda(\mathbf{Z}^n)$$

  where $|u_i| = 1$.