1 Poincaré Duality

1.1 Orientation in manifolds

- $R$ a ring (usually $R = \mathbb{Z}$ or $R = \mathbb{Z}_2$).
- $M$ an $n$-manifold.
- Open ball $B$ of $M$ is image under some chart $\phi : \mathbb{R}^n \to M$ of a ball of radius $\varepsilon > 0$.
- Open balls give basis for topology on $M$.
- For all $x \in M$ let $B$ be an open ball neighborhood of $x$. By excising $M - B$

$$H_n(M|x;R) \cong H_n(B,B - \{x\};R)$$

$$\cong \tilde{H}_{n-1}(B,B - \{x\};R)$$

$$\cong \tilde{H}_{n-1}(S^{n-1};R)$$

$$\cong R$$

**Definition 1.1** (Local orientation). A **local $R$-orientation** at $x \in M$ is a choice of unit

$$\mu_x \in H_n(M,M - \{x\};R) \cong R$$

**Definition 1.2** (Local homology and localization).

- Fix a topological space $X$ and abelian group $G$.
- For $A \subset X$ the **local homology** at $A$ with coefficients in $G$ is

$$H_k(X|A;G) := H(X,X - A;G)$$

- Similarly if $x \in X$ the **local homology** at $x$ with coefficients in $G$ is

$$H_k(X|x;G) := H(X,X - \{x\};G)$$

- If $A \subset B \subset X$ then we have the inclusion map of topological pairs

$$i^B : (X,X - B) \to (X,X - A)$$

which induces the **localization** to $A$

$$i^A_* : H_k(X|B;G) \to H_k(X|A;G)$$
\[
\begin{align*}
\text{• Note that the domain of } i_x^\alpha \text{ is ambiguous but should be clear from context.}
\end{align*}
\]

Let \( \alpha : M \to \coprod_{x \in M} H_n(M|x; R) \) be a function such that
\[
\alpha(x) \in H_n(M|x; R) \cong R
\]

**Definition 1.3** (Locally consistent). \( \alpha \) is **locally consistent** if for all \( x \in M \) there is an open ball neighborhood \( B \subset M \) of \( x \) and \( \alpha_B \in H_n(M|B; R) \) such that for all \( y \in B \) the inclusion map \( i_y^\alpha : (M, M - \{y\}) \to (M, M - \{y\}) \) induces \( i_y^\alpha : H_n(M|B; R) \to H_n(M|y; R) \) satisfying
\[
i_y^\alpha(\alpha_B) = \alpha(y)
\]

**Definition 1.4** (Orientation). An **R-orientation** on the manifold \( M \) is a function \( \mu : M \to \coprod_{x \in M} H_n(M|x; R) \) with
\[
\mu(x) \in H_n(M|x; R) \cong R
\]
a unit for all \( x \in M \) which furthermore is locally consistent.

**\( R \)-module of locally consistent functions**

- Set
\[
\Gamma_R(M) = \left\{ \alpha \mid \alpha : M \to \prod_{x \in M} H_n(M|x; R) \text{ locally consistent} \right\}
\]
to be the set of locally consistent functions on \( M \).
- Sums and scalar multiples of locally consistent functions are locally consistent so \( \Gamma_R(M) \) is an \( R \)-module.
- Let \( \ell : H_n(M; R) \to \Gamma_R(M) \) be \( R \)-module homomorphism where for \( \alpha_M \in H_n(M; R) = H_n(M|M; R) \)
we set \( \ell(\alpha_M) \in \Gamma_R(M) \) to be the function
\[
\ell(\alpha_M)(x) = i_x^\alpha(\alpha_M) \quad \text{for } x \in M.
\]
- We will show \( \ell \) is an isomorphism for closed \( R \)-orientable \( M \).

**Definition 1.5.**
\[
M_R = \left\{ (x, \alpha_x) \mid x \in M \text{ and } \alpha_x \in H_n(M|x; R) \right\}
\]
For each open ball \( B \subset M \) and \( \alpha_B \in H_n(M|B; R) \cong R \) set
\[
U(B, \alpha_B) = \left\{ (x, \alpha_x) \mid x \in B \text{ and } \alpha_x = i_x^\alpha(\alpha_B) \right\}
\]
Topologize \( M_R \) using basis
\[
\left\{ U(B, \alpha_B) \mid B \text{ open ball of } M \text{ and } \alpha_B \in H_n(M|B; R) \right\}
\]
The map \( p : M_R \to M \) with \( p(x, \alpha_x) = x \) is a covering map.

A **section** of \( M_R \) is a continuous map \( s : M \to M_R \) with \( p(s(x)) = x \) for all \( x \in M \). Note that sections of \( M_R \) correspond 1-1 with locally consistent functions \( \alpha : M \to \coprod_{x \in M} H_n(M, M - \{x\}; R) \) via the bijection \( s \mapsto \alpha \) with \( s(x) = (x, \alpha(x)) \).

The following lemma allows us to “integrate” our locally described orientation (or any other locally consistent function) to get a unique homology class on an arbitrarily large compact subset \( K \subset M \).

**Lemma 1.6.** \( M \) an \( n \)-manifold and \( K \subset M \) compact. Then
1. $H_k(M|K; R) = 0$ for $k > n$.

2. $\alpha_K \in H_k(M|K; R)$ is 0 if and only if $i_*^\gamma(\alpha_K) = 0$ for all $x \in K$.

3. If $\alpha : M \to \bigcup_{x \in M} H_n(M|x; R)$ is locally consistent then there is unique $\alpha_K \in H_n(M|K; R)$ such that $i_*^\gamma(\alpha_K) = \alpha(x)$ for all $x \in K$.

Proof.

• Suppose:
  1. $M$ is an $n$-manifold
  2. $\alpha : M \to \bigcup_{x \in M} H_n(M|x; R)$ is locally consistent.

• First we show:

Claim I: If Lemma holds for compact subsets $A$, $B$ and $A \cap B$ in $M$ then it holds for $A \cup B$.

• Suppose Lemma holds for compact subsets $A$, $B$ and $A \cap B$ in $M$.

• Then for $k > n$ the relative Mayer-Vietoris sequence yields exact

$$H_k(M|A \cap B; R) \to H_k(M|A \cup B; R) \to H_k(M|A; R) \oplus H_k(M|B; R)$$

Hence $H_k(M|A \cup B; R) = 0$ for $k > n$.

• Thus Part 1 of Lemma holds for $A \cup B$.

• Again relative Mayer-Vietoris sequence yields exact

$$0 \to H_n(M|A \cup B; R) \xrightarrow{\Phi} H_n(M|A; R) \oplus H_n(M|B; R) \xrightarrow{\Psi} H_n(M|A \cap B; R)$$

where $\Phi(\gamma) = (i_*^A(\gamma), -i_*^B(\gamma))$ and $\Psi(\gamma, \eta) = i_*^{A\cap B}(\gamma) + i_*^{A\cup B}(\eta)$.

• By assumption we have unique classes

$$\alpha_A \in H_n(M|A; R)$$
$$\alpha_B \in H_n(M|B; R)$$
$$\alpha_{A\cap B} \in H_n(M|A \cap B; R)$$

such that

$$\alpha(x) = i_*^\gamma(\alpha_A) \text{ for all } x \in A$$
$$\alpha(x) = i_*^\gamma(\alpha_B) \text{ for all } x \in B$$
$$\alpha(x) = i_*^\gamma(\alpha_{A\cap B}) \text{ for all } x \in A \cap B$$

• Note that for all $x \in A \cap B$

$$i_*^{\gamma_X}_{i_*^\gamma}(\alpha_A) = i_*^\gamma(\alpha_A) = \alpha(x)$$
$$i_*^{\gamma_X}_{i_*^\gamma}(\alpha_B) = i_*^\gamma(\alpha_B) = \alpha(x)$$

Thus both $i_*^{A\cap B}(\alpha_A)$ and $i_*^{A\cap B}(\alpha_B)$ must be the unique class $H_n(M|A \cap B; R)$ with this property.
• Hence \( i_*^{A \cap B}(\alpha_A) = i_*^{A \cap B}(\alpha_B) = \alpha_{A \cap B} \).

• and \( \Psi(\alpha_A, -\alpha_B) = i_*^{A \cap B}(\alpha_A) - i_*^{A \cap B}(\alpha_B) = \alpha_{A \cap B} - \alpha_{A \cap B} = 0 \).

• By exactness of

\[
0 \rightarrow H_n(M|A \cup B; R) \xrightarrow{\Phi} H_n(M|A; R) \oplus H_n(M|B; R) \xrightarrow{\Psi} H_n(M|A \cap B; R)
\]

there is unique \( \alpha_{A \cup B} \in H_n(M|A \cap B; R) \) such that \( \Phi(\alpha_{A \cup B}) = (\alpha_A, -\alpha_B) \).

• If \( x \in A \) then \( i_*^x(\alpha_{A \cup B}) = i_*^x(i_*^A(\alpha_{A \cup B})) = i_*^x(\alpha_A) = \alpha(x) \).

• If \( x \in B \) then \( i_*^x(\alpha_{A \cup B}) = i_*^x(i_*^B(\alpha_{A \cup B})) = i_*^x(\alpha_B) = \alpha(x) \).

• Hence if \( x \in A \cup B \) then \( i_*^x(\alpha_{A \cup B}) = \alpha(x) \).

• By injectivity of \( \Phi \) any other class \( \gamma \in H_n(M|A \cap B; R) \) must have either \( i_*^A(\gamma) \neq \alpha_A \) or \( i_*^B(\gamma) \neq \alpha_B \).

• Suppose \( i_*^A(\gamma) \neq \alpha_A \)

• Then by uniqueness of \( \alpha_A \) must be some \( x \in A \) such that

\[
i_*^x(\gamma) = i_*^x(i_*^A(\gamma)) \neq \alpha(x)
\]

• Same argument for \( i_*^B(\gamma) \neq \alpha_B \) shows uniqueness of \( \alpha_{A \cup B} \).

• Lemma holds for \( A \cup B \).

• Thus Claim I holds.

Claim II: If Lemma holds for \( M = \mathbb{R}^n \) then it holds for all manifolds

• Suppose \( M \) is an \( n \)-manifold and \( K \subset M \) is compact.

• Suppose Lemma holds for all compact \( K' \subset \mathbb{R}^n \).

• We may write \( K = K_1 \cup \cdots \cup K_m \) with each \( K_i \subset \phi_i(\mathbb{R}^n) \subset M \) compact.

• Let \( A = K_m \) and \( B = K_1 \cup \cdots \cup K_{m-1} \). Then \( A \cup B = K \) and

\[
A \cap B = (K_1 \cap K_m) \cup \cdots \cup (K_{m-1} \cap K_m)
\]

is union of \( m - 1 \) compact sets in \( \mathbb{R}^n \).

• Lemma holds for \( A \) by assumption.

• By inductive assumption Lemma holds for \( B \) and \( A \cap B \).

• Hence by above, Lemma holds for \( K = A \cup B \).

Claim III: If \( M = \mathbb{R}^n \) and \( K \subset M \) is finite union of convex compact sets then the Lemma holds.

• By inductive argument from Claim II only need case where \( K \) is convex.

• Choose a large open ball \( B \subset \mathbb{R}^n \) so that \( K \subset B \). For all \( x \in K \) have a deformation retracts of \( M - K \) and \( M - \{x\} \) to \( S^{n-1} \) so

\[
H_n(M|B; R) \cong H_n(M|K; R) \cong H_n(M|x; R)
\]

Claim IV: If \( M = \mathbb{R}^n \) and \( K \subset \mathbb{R}^n \) is compact then the Lemma holds.

• revise

\[\square\]