# Algebraic Topology II – Lecture 17

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# 1 Poincaré Duality

## 1.1 Orientation in manifolds

- $R$ a ring (usually $R = \mathbf{Z}$ or $R = \mathbf{Z}_2$).
- $M$ an $n$-manifold.
- **Open ball** $B$ of $M$ is image under some chart $\phi : \mathbf{R}^n \to M$ of a ball of radius $\varepsilon > 0$.
- Open balls give basis for topology on $M$.
- For all $x \in M$ let $B$ be an open ball neighborhood of $x$. By excising $M - B$

$$\begin{aligned} H_n(M|x;R) &\cong H_n(B, B - \{x\}; R) \\ &\cong \widetilde{H}_{n-1}(B, B - \{x\}; R) \\ &\cong \widetilde{H}_{n-1}(S^{n-1}; R) \\ &\cong R \end{aligned}$$

**Definition 1.1** (Local orientation)**.** A **local $R$-orientation** at $x \in M$ is a choice of unit

$$\mu_x \in H_n(M, M - \{x\}; R) \cong R$$

**Definition 1.2** (Local homology and localization)**.**

- Fix a topological space $X$ and abelian group $G$.
- For $A \subset X$ the **local homology** at $A$ with coefficients in $G$ is

$$H_k(X|A;G) := H(X, X - A; G)$$

- Similarly if $x \in X$ the **local homology** at $x$ with coefficients in $G$ is

$$H_k(X|x;G) := H(X, X - \{x\}; G)$$

- If $A \subset B \subset X$ then we have the inclusion map of topological pairs

$$i^B : (X, X - B) \to (X, X - A)$$

which induces the **localization** to $A$

$$i^A_* : H_k(X|B;G) \to H_k(X|A;G)$$

- Note that the domain of $i^A_*$ is ambiguous but should be clear from context.

Let $\alpha : M \to \coprod_{x \in M} H_n(M|x;R)$ be a function such that

$$\alpha(x) \in H_n(M|x;R) \cong R$$

**Definition 1.3** (Locally consistent)**.** $\alpha$ is **locally consistent** if for all $x \in M$ there is an open ball neighborhood $B \subset M$ of $x$ and $\alpha_B \in H_n(M|B;R)$ such that for all $y \in B$ the inclusion map $i^y : (M, M-B) \to (M, M-\{y\})$ induces $i^y_* : H_n(M|B;R) \to H_n(M|y;R)$ satisfying

$$i^y_*(\alpha_B) = \alpha(y)$$

**Definition 1.4** (Orientation)**.** An $R$**-orientation** on the manifold $M$ is a function $\mu : M \to \coprod_{x \in M} H_n(M|x;R)$ with

$$\mu(x) \in H_n(M|x;R) \cong R$$

a unit for all $x \in M$ which furthermore is locally consistent.

**$R$-module of locally consistent functions**

- Set
$$\Gamma_R(M) = \left\{ \alpha \,\middle|\, \alpha : M \to \coprod_{x \in M} H_n(M|x;R) \text{ locally consistent} \right\}$$
to be the set of locally consistent functions on $M$.

- Sums and scalar multiples of locally consistent functions are locally consistent so $\Gamma_R(M)$ is an $R$-module.

- Let $\ell : H_n(M;R) \to \Gamma_R(M)$ be $R$-module homomorphism where for $\alpha_M \in H_n(M;R) = H_n(M|M;R)$ we set $\ell(\alpha_M) \in \Gamma_R(M)$ to be the function
$$\ell(\alpha_M)(x) = i^x_*(\alpha_M) \qquad \text{for } x \in M.$$

- We will show $\ell$ is an isomorphism for closed $R$-orientable $M$.

**Definition 1.5.**

$$M_R \underset{\text{setwise}}{=} \left\{ (x, \alpha_x) \,\middle|\, x \in M \text{ and } \alpha_x \in H_n(M|x;R) \right\}$$

For each open ball $B \subset M$ and $\alpha_B \in H_n(M|B;R) \cong R$ set

$$U(B, \alpha_B) = \left\{ (x, \alpha_x) \,\middle|\, x \in B \text{ and } \alpha_x = i^x_*(\alpha_B) \right\}$$

Topologize $M_R$ using basis

$$\left\{ U(B, \alpha_B) \,\middle|\, B \text{ open ball of } M \text{ and } \alpha_B \in H_n(M|B;R) \right\}$$

The map $p : M_R \to M$ with $p(x, \alpha_x) = x$ is a covering map.

A **section** of $M_R$ is a continuous map $s : M \to M_R$ with $p(s(x)) = x$ for all $x \in M$. Note that sections of $M_R$ correspond 1-1 with locally consistent functions $\alpha : M \to \coprod_{x \in M} H_n(M, M - \{x\}; R)$ via the bijection $s \mapsto \alpha$ with $s(x) = (x, \alpha(x))$

The following lemma allows us to "integrate" our locally described orientation (or any other locally consistent function) to get a unique homology class on an arbitrarily large compact subset $K \subset M$.

**Lemma 1.6.** *$M$ an $n$-manifold and $K \subset M$ compact. Then*

1. *$H_k(M|K;R) = 0$ for $k > n$.*
2. *$\alpha_K \in H_k(M|K;R)$ is 0 if and only if $i_*^x(\alpha_K) = 0$ for all $x \in K$.*
3. *If $\alpha : M \to \coprod_{x\in M} H_n(M|x;R)$ is locally consistent then there is unique*

$$\alpha_K \in H_n(M|K;R)$$

*such that $i_*^x(\alpha_K) = \alpha(x)$ for all $x \in K$.*

*Proof.*

- Suppose:
  1. $M$ is an $n$-manifold
  2. $\alpha : M \to \coprod_{x\in M} H_n(M|x;R)$ is locally consistent.
- First we show:

**Claim I:** *If Lemma holds for compact subsets $A$, $B$ and $A \cap B$ in $M$ then it holds for $A \cup B$.*

- Suppose Lemma holds for compact subsets $A$, $B$ and $A \cap B$ in $M$.
- Then for $k > n$ the relative Mayer-Vietoris sequence yields exact

  $$\underbrace{H_k(M|A\cap B;R)}_{0} \to H_k(M|A\cup B;R) \to \underbrace{H_k(M|A;R)}_{0} \oplus \underbrace{H_k(M|B;R)}_{0}$$

  Hence $H_k(M|A\cup B;R) = 0$ for $k > n$.
- Thus Part 1 of Lemma holds for $A \cup B$.
- Again relative Mayer-Vietoris sequence yields exact

  $$0 \to H_n(M|A\cup B;R) \xrightarrow{\Phi} H_n(M|A;R) \oplus H_n(M|B;R) \xrightarrow{\Psi} H_n(M|A\cap B;R)$$

  where $\Phi(\gamma) = (i_*^A(\gamma), -i_*^B(\gamma))$ and $\Psi(\gamma,\eta) = i_*^{A\cap B}(\gamma) + i_*^{A\cap B}(\eta)$.
- By assumption we have unique classes

  $$\alpha_A \in H_n(M|A;R)$$
  $$\alpha_B \in H_n(M|B;R)$$
  $$\alpha_{A\cap B} \in H_n(M|A\cap B;R)$$

  such that

  $$\alpha(x) = i_*^x(\alpha_A) \text{ for all } x \text{ in } A$$
  $$\alpha(x) = i_*^x(\alpha_B) \text{ for all } x \text{ in } B$$
  $$\alpha(x) = i_*^x(\alpha_{A\cap B}) \text{ for all } x \text{ in } A\cap B$$

- Note that for all $x \in A \cap B$

  $$i_*^x i_*^{A\cap B}(\alpha_A) = i_*^x(\alpha_A) = \alpha(x)$$
  $$i_*^x i_*^{A\cap B}(\alpha_B) = i_*^x(\alpha_B) = \alpha(x)$$

  Thus both $i_*^{A\cap B}(\alpha_A)$ and $i_*^{A\cap B}(\alpha_B)$ must be the unique class $H_n(M|A\cap B;R)$ with this property.

- Hence $i_*^{A\cap B}(\alpha_A) = i_*^{A\cap B}(\alpha_B) = \alpha_{A\cap B}$.
- and $\Psi(\alpha_A, -\alpha_B) = i_*^{A\cap B}(\alpha_A) - i_*^{A\cap B}(\alpha_B) = \alpha_{A\cap B} - \alpha_{A\cap B} = 0$.
- By exactness of

$$0 \to H_n(M|A \cup B; R) \xrightarrow{\Phi} H_n(M|A; R) \oplus H_n(M|B; R) \xrightarrow{\Psi} H_n(M|A \cap B; R)$$

  there is unique $\alpha_{A\cup B} \in H_n(M|A \cap B; R)$ such that $\Phi(\alpha_{A\cup B}) = (\alpha_A, -\alpha_B)$.
- If $x \in A$ then $i_*^x(\alpha_{A\cup B}) = i_*^x(i_*^A(\alpha_{A\cup B})) = i_*^x(\alpha_A) = \alpha(x)$.
- If $x \in B$ then $i_*^x(\alpha_{A\cup B}) = i_*^x(i_*^B(\alpha_{A\cup B})) = i_*^x(\alpha_B) = \alpha(x)$
- Hence if $x \in A \cup B$ then $i_*^x(\alpha_{A\cup B}) = \alpha(x)$.
- By injectivity of $\Phi$ any other class $\gamma \in H_n(M|A \cap B; R)$ must have either $i_*^A(\gamma) \neq \alpha_A$ or $i_*^B(\gamma) \neq \alpha_B$.
- Suppose $i_*^A(\gamma) \neq \alpha_A$
- Then by uniqueness of $\alpha_A$ must be some $x \in A$ such that

$$i_*^x(\gamma) = i_*^x(i_*^A(\gamma)) \neq \alpha(x)$$

- Same argument for $i_*^B(\gamma) \neq \alpha_B$ shows uniqueness of $\alpha_{A\cup B}$.
- Lemma holds for $A \cup B$.
- Thus Claim I holds.

**Claim II:** *If Lemma holds for $M = \mathbf{R}^n$ then it holds for all manifolds*

- Suppose $M$ is an $n$-manifold and $K \subset M$ is compact.
- Suppose Lemma holds for all compact $K' \subset \mathbf{R}^n$.
- We may write $K = K_1 \cup \cdots \cup K_m$ with each $K_i \subset \phi_i(\mathbf{R}^n) \subset M$ compact.
- Let $A = K_m$ and $B = K_1 \cup \cdots \cup K_{m-1}$. Then $A \cup B = K$ and

$$A \cap B = (K_1 \cap K_m) \cup \cdots \cup (K_{m-1} \cap K_m)$$

  is union of $m - 1$ compact sets in $\mathbf{R}^n$.
- Lemma holds for $A$ by assumption.
- By inductive assumption Lemma holds for $B$ and $A \cap B$.
- Hence by above, Lemma holds for $K = A \cup B$.

**Claim III:** *If $M = \mathbf{R}^n$ and $K \subset M$ is finite union of convex compact sets then the Lemma holds.*

- By inductive argument from Claim II only need case where $K$ is convex.
- Choose a large open ball $B \subset \mathbf{R}^n$ so that $K \subset B$. For all $x \in K$ have a deformation retracts of $M - K$ and $M - \{x\}$ to $S^{n-1}$ so
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$$H_n(M|B; R) \cong H_n(M|K; R) \cong H_n(M|x; R)$$

**Claim IV:** *If $M = \mathbf{R}^n$ and $K \subset \mathbf{R}^n$ is compact then the Lemma holds.*

- *revise*

□