1 Poincaré Duality

1.1 Orientation in manifolds

Lemma 1.1. \(\text{M an}\) \(n\)-manifold and \(K \subset M\) compact. Then

1. \(H_k(M|K;R) = 0\) for \(k > n\).
2. \(\alpha_K \in H_k(M|K;R)\) is 0 if and only if \(i_x^\ast(\alpha_K) = 0\) for all \(x \in K\).
3. If \(\alpha : M \to \bigsqcup_{x \in M} H_n(M|x;R)\) is locally consistent then there is unique

\[\alpha_K \in H_n(M|K;R)\]

such that \(i_x^\ast(\alpha_K) = \alpha(x)\) for all \(x \in K\).

Theorem 1.2. \(\text{M a closed connected}\) \(n\)-manifold. Then

1. If \(M\) is \(R\)-orientable then the map \(i_x^\ast : H_n(M;R) \to H_n(M|x;R)\) is an isomorphism for all \(x \in M\).
2. If \(M\) is not \(R\)-orientable then the map \(i_x^\ast : H_n(M;R) \to H_n(M|x;R)\) is injective with image \(\{r \in R|2r = 0\}\) for all \(x \in M\).
3. \(H_k(M;R) = 0\) for \(k > n\).

Proof.

- If \(M\) is closed then it is compact.
- Apply Lemma 1.1 Part 1 with \(M = A\). If \(k > n\) then
  \[H_k(M;R) = H_k(M|M;R) = 0.\]
- Let \(\Gamma_R(M)\) be the set of locally consistent functions \(\alpha : M \to \bigsqcup_{x \in M} H_n(M|x;R)\).
- Sums and scalar multiples of locally consistent functions are locally consistent.
- Thus \(\Gamma_R(M)\) is an \(R\)-module.
- Let \(\gamma : H_n(M;R) \to \Gamma_R(M)\) be \(R\)-module homomorphism where for \(\alpha_M \in H_n(M;R) = H_n(M|M;R)\)
  we set locally consistent \(\gamma(\alpha_M) : M \to \bigsqcup_{x \in M} H_n(M|x;R)\) to be
  \[\gamma(\alpha_M)(x) = i_x^\ast(\alpha_M)\]
  for \(x \in M\).
- Lemma 1.1 Part 2 says \(\gamma\) is injective.
Lemma 1.1 Part 3 says \( \gamma \) is surjective if \( M \) is \( R \)-orientable.

\[ \square \]

**Definition 1.3** (Fundamental class). For an \( n \)-manifold \( M \) a **fundamental class** is a class \( \mu \in H_n(M; R) \) such that \( i^*_\gamma(\mu) \) is a generator of \( H_n(M; R; R) \) for all \( x \in M \).

- By Theorem 1.2 above, every closed \( R \)-orientable manifold has a fundamental class.
- A fundamental class gives an \( R \)-orientation for \( M \) so \( R \)-orientability is necessary in order to have a fundamental class.
- A cycle representing an element of \( H_n(M; R) \) has compact image and so can induce nonzero local homology on only a compact subset of \( M \) hence compactness is necessary in order to have a fundamental class.

### 1.2 Slant and cap products

For spaces \( X, Y \) and abelian groups \( G_1, G_2 \) define

\[ -\partial : C^p(Y; G_1) \otimes (C(X) \otimes C(Y))_n \otimes G_2 \rightarrow (C_{n-p}(X) \otimes G_1) \otimes G_2 \]

where for \( u \in C^p(Y; G_1) \), \( \sigma \in C_q(X) \), \( \eta \in C_r(Y) \) and \( g_2 \in G_2 \) we set

\[ u\partial (\sigma \otimes \eta \otimes g_2) = \sigma \otimes u(\eta) \otimes g_2. \]

**Definition 1.4** (Slant product of (co)chains). \( X \) and \( Y \) spaces and \( G_1 \) and \( G_2 \) abelian groups with pairing \( \omega : G_1 \otimes G_2 \rightarrow G_3 \). The **slant product** is the pairing

\[ -\partial : C^p(Y; G_1) \otimes C_n(X \times Y; G_2) \rightarrow C_{n-p}(X; G_1 \otimes G_2) \]

where for \( u \in C^p(Y; G_1) \), \( \sigma \in C_n(X \times Y) \) and \( g_2 \in G_2 \) we set

\[ u\partial (\sigma \otimes g_2) = (1_{C(X)} \otimes \omega)(u\partial [(\omega \sigma) \otimes g_2]) \]

Where \( A : C(X \times Y) \rightarrow C(X) \otimes C(Y) \) is the Alexander-Whitney map.

We can simplify the formula for the slant product of \( u \in C^p(Y; G_1) \) and \( \sigma \otimes g_2 \in C_n(X \times Y; G_2) \) as follows

\[ u\partial (\sigma \otimes g_2) = (1_{C(X)} \otimes \omega)(u\partial [(\omega \sigma) \otimes g_2]) \]

\[ = (1_{C(X)} \otimes \omega) \sum_{k=0}^{n} u\partial \left( (\pi_X \circ \sigma \circ f_k) \otimes (\pi_Y \circ \sigma \circ b_{n-k}) \otimes g_2 \right) \]

\[ = (1_{C(X)} \otimes \omega) \sum_{k=0}^{n} (\pi_X \circ \sigma \circ f_k) \otimes u(\pi_Y \circ \sigma \circ b_{n-k}) \otimes g_2 \]

\[ = (1_{C(X)} \otimes \omega) \left( \pi_X \circ \sigma \circ f_{n-p} \right) \otimes u(\pi_Y \circ \sigma \circ b_{p}) \otimes g_2 \]

\[ = (\pi_X \circ \sigma \circ f_{n-p}) \otimes \omega \left( u(\pi_Y \circ \sigma \circ b_{p}) \otimes g_2 \right) \]
Lemma 1.5 (Slant product and (co)boundaries). If $u \in C^p(Y; G_1)$, $\sigma \in C_q(X)$, $\eta \in C_r(Y)$ and $g_2 \in G_2$ then
$$\partial (u \delta (\sigma \otimes \eta \otimes g_2)) = (\delta u) \delta (\sigma \otimes \eta \otimes g_2) + (-1)^p u \partial (\sigma \otimes \eta \otimes g_2).$$
If $u \in C^p(Y; G_1)$ and $\sigma \in C_q(X; G_2)$ then
$$\partial (u \eta \delta ) = (\delta u) \eta \delta + (-1)^p u \partial \eta \delta.$$

As a simple corollary we see that the slant product on (co)chains induces a well-defined slant product on (co)homology.

Definition 1.6 (Slant product on (co)homology). The slant product on (co)homology is the pairing
$$\cap^* : H^p(Y; G_1) \otimes H_n(X \times Y; G_2) \to H_{n-p}(X; G_3)$$
where for cocycle $u \in C^p(Y; G_1)$ and cycle $\sigma \in C_n(X \times Y; G_2)$ we set
$$[u] \cap [\sigma] = [u \cap \sigma].$$

Lemma 1.7 (Naturality of slant product). For maps $f : X \to W$ and $g : Y \to Z$ cochain $u \in C^p(Z; G_1)$ and chain $\sigma \in C_n(X \times Y; G_2)$ we have
$$f_* ((g^* u) \cap \sigma) = u \cap (f \times g)_* \sigma$$
which induces the corresponding equality on (co)homology groups.

Note this is not quite our formal version of naturality

$$\begin{array}{ccc}
C^p(Y; G_1) \otimes C_n(X \times Y; G_2) & \xrightarrow{\cap^*} & H_{n-p}(X; G_3) \\
g^* & \uparrow & \uparrow (f \times g)_* \\
C^p(Z; G_1) \otimes C_n(W \times Z; G_2) & \xrightarrow{\cap^*} & H_{n-p}(Z; G_3)
\end{array}$$

Definition 1.8 (Cap product). $X$ a space $G_1$ and $G_2$ abelian groups with pairing $\omega : G_1 \otimes G_2 \to G_3$. The cap product is the pairing
$$\cap^- : C^p(X; G_1) \otimes C_n(X; G_2) \to C_{n-p}(X; G_3)$$
where for $u \in C^p(X; G_1)$, $\sigma \in C_n(X; G_2)$ we set
$$u \cap \sigma = u \Delta \sigma.$$

where $\Delta : X \to X \times X$ is the diagonal map.

This induces the cap product on (co)homology
$$\cap^- : H^p(X; G_1) \otimes H_n(X; G_2) \to H_{n-p}(X; G_3).$$

This formula simplifies to
$$u \cap (\sigma \otimes g_2) = (\pi_X \circ \Delta \sigma \circ f_{n-p}) \otimes \omega \left( u (\pi_X \circ \Delta \sigma \circ b_p) \otimes g_2 \right)$$
$$= (\sigma \circ f_{n-p}) \otimes \omega \left( u (\sigma \circ b_p) \otimes g_2 \right)$$

Naturality of the slant product gives us naturality for cap product:
Corollary 1.9 (Naturality of cap product). For map $f : X \to Y$ cochain $u \in C^p(Z; G_1)$ and chain $\sigma \in C_n(X \times Y; G_2)$ we have

$$f_\ast((f_\ast u) \smile \sigma) = u \smile f_\ast \sigma$$

which induces the corresponding equality on (co)homology groups.

Note this is not quite our formal version of naturality

\[
\begin{array}{ccc}
C^p(X; G_1) \otimes C_n(X; G_2) & \longrightarrow & C_{n-p}(X; G_3) \\
\downarrow f_\ast & & \downarrow f_\ast \\
C^p(Y; G_1) \otimes C_n(Y; G_2) & \longrightarrow & C_{n-p}(Y; G_3)
\end{array}
\]

1.3 Cohomology with compact supports

In the case we are most interested where $G_1 = G_2 = R$ with the multiplication pairing we get the formula

$$u \smile r\sigma = \sigma \circ f_{n-q} \otimes u(\sigma \circ b_p) \cdot r$$

Theorem 1.10 (Poincaré Duality). $M$ an $R$-orientable $n$-manifold with fundamental class $[M] \in H_n(M; R)$. Then

$$- \smile [M] : H^p(X; R) \to H_{n-p}(X; R)$$

is an isomorphism for all $p$.

Recall that $k$-cochains of $X$ are functions

$$C^k(X; G) = G^{S_k(X)}$$

where $S_k(X)$ is set of singular simplices of $X$.

Definition 1.11 (Cohomology with compact supports). $X$ a space $G$ abelian. $k$-cochains with compact support are:

$$C^k_c(X; G) = \left\{ f \in C^k(X; G) \, \middle| \, \exists \text{ compact } K \subset X \text{ s.t. } f(\sigma) = 0 \text{ for } \sigma \in S_k(X - K; G) \right\}$$

Cohomology with compact supports of $X$ is cohomology $H^*_c(X; G)$ of this cochain complex.