Algebraic Topology II – Lecture 18

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1 Poincaré Duality

1.1 Orientation in manifolds

Lemma 1.1. *M an n-manifold and $K \subset M$ compact. Then*

1. $H_k(M|K;R) = 0$ *for* $k > n$.
2. $\alpha_K \in H_k(M|K;R)$ *is 0 if and only if* $i^x_*(\alpha_K) = 0$ *for all* $x \in K$.
3. *If* $\alpha : M \to \coprod_{x\in M} H_n(M|x;R)$ *is locally consistent then there is unique*

$$\alpha_K \in H_n(M|K;R)$$

such that $i^x_*(\alpha_K) = \alpha(x)$ *for all* $x \in K$.

Theorem 1.2. *M a closed connected n-manifold. Then*

1. *If M is R-orientable then the map* $i^x_* : H_n(M;R) \to H_n(M|x;R)$ *is an isomorphism for all* $x \in M$.
2. *If M is not R-orientable then the map* $i^x_* : H_n(M;R) \to H_n(M|x;R)$ *is injective with image* $\{r \in R|\, 2r = 0\}$ *for all* $x \in M$.
3. $H_k(M;R) = 0$ *for* $k > n$.

Proof.

- If M is closed then it is compact.
- Apply Lemma 1.1 Part 1 with $M = A$. If $k > n$ then

$$H_k(M;R) = H_k(M|M;R) = 0.$$

- Let $\Gamma_R(M)$ be the set of locally consistent functions $\alpha : M \to \coprod_{x\in M} H_n(M|x;R)$.
- Sums and scalar multiples of locally consistent functions are locally consistent.
- Thus $\Gamma_R(M)$ is an R-module.
- Let $\gamma : H_n(M;R) \to \Gamma_R(M)$ be R-module homomorphism where for $\alpha_M \in H_n(M;R) = H_n(M|M;R)$ we set locally consistent $\gamma(\alpha_M) : M \to \coprod_{x\in M} H_n(M|x;R)$ to be

$$\gamma(\alpha_M)(x) = i^x_*(\alpha_M) \qquad \text{for } x \in M.$$

- Lemma 1.1 Part 2 says γ is injective.

- Lemma 1.1 Part 3 says γ is surjective if M is R-orientable.

$\square$

Definition 1.3 (Fundamental class)**.** For an n-manifold M a **fundamental class** is a class $\mu \in H_n(M;R)$ such that $i_*^x(\mu)$ is a generator of $H_n(M|x;R)$ for all $x \in M$.

- By Theorem 1.2 above, every closed R-orientable manifold has a fundamental class.
- A fundamental class gives an R-orientation for M so R-orientability is necessary in order to have a fundamental class.
- A cycle representing an element of $H_n(M;R)$ has compact image and so can induce nonzero local homology on only a compact subset of M hence compactness is necessary in order to have a fundamental class.

1.2 Slant and cap products

For spaces X, Y and abelian groups G_1, G_2 define

$$-\backslash\!\!\backslash- : C^p(Y;G_1) \otimes \Big([C(X) \otimes C(Y)]_n \otimes G_2\Big) \to \Big(C_{n-p}(X) \otimes G_1 \otimes G_2\Big)$$

where for $u \in C^p(Y;G_1)$, $\sigma \in C_q(X)$, $\eta \in C_r(Y)$ and $g_2 \in G_2$ we set

$$u \backslash\!\!\backslash (\sigma \otimes \eta \otimes g_2) = \sigma \otimes u(\eta) \otimes g_2.$$

Definition 1.4 (Slant product of (co)chains)**.** X and Y spaces and G_1 and G_2 abelian groups with pairing $\omega : G_1 \otimes G_2 \to G_3$. The **slant product** is the pairing

$$-\backslash- : C^p(Y;G_1) \otimes C_n(X \times Y;G_2) \to C_{n-p}(X;G_1 \otimes G_2)$$

where for $u \in C^p(Y;G_1)$, $\sigma \in C_n(X \times Y)$ and $g_2 \in G_2$ we set

$$u\backslash(\sigma \otimes g_2) = (\mathbf{1}_{C(X)} \otimes \omega)(u \backslash\!\!\backslash [(A\sigma) \otimes g_2])$$

Where $A : C(X \times Y) \to C(X) \otimes C(Y)$ is the Alexander-Whitney map.

We can simplify the formula for the slant product of $u \in C^p(Y;G_1)$ and $\sigma \otimes g_2 \in C_n(X \times Y;G_2)$ as follows

$$\begin{aligned}
u\backslash(\sigma \otimes g_2) &= (\mathbf{1}_{C(X)} \otimes \omega)(u \backslash\!\!\backslash [(A\sigma) \otimes g_2]) \\
&= (\mathbf{1}_{C(X)} \otimes \omega) \sum_{k=0}^{n} u \backslash\!\!\backslash \Big((\pi_X \circ \sigma \circ f_k) \otimes (\pi_Y \circ \sigma \circ b_{n-k}) \otimes g_2\Big) \\
&= (\mathbf{1}_{C(X)} \otimes \omega) \sum_{k=0}^{n} (\pi_X \circ \sigma \circ f_k) \otimes u(\pi_Y \circ \sigma \circ b_{n-k}) \otimes g_2 \\
&= (\mathbf{1}_{C(X)} \otimes \omega)\Big((\pi_X \circ \sigma \circ f_{n-p}) \otimes u(\pi_Y \circ \sigma \circ b_p) \otimes g_2\Big) \\
&= (\mathbf{1}_{C(X)}\, \pi_X \circ \sigma \circ f_{n-p}) \otimes \omega\Big(u(\pi_Y \circ \sigma \circ b_p) \otimes g_2\Big) \\
&= (\pi_X \circ \sigma \circ f_{n-p}) \otimes \omega\Big(u(\pi_Y \circ \sigma \circ b_p) \otimes g_2\Big)
\end{aligned}$$

Lemma 1.5 (Slant product and (co)boundaries)**.** *If $u \in C^p(Y; G_1)$, $\sigma \in C_q(X)$, $\eta \in C_r(Y)$ and $g_2 \in G_2$ then*

$$\partial(u \backslash\!\!\backslash (\sigma \otimes \eta \otimes g_2)) = (\delta u) \backslash\!\!\backslash (\sigma \otimes \eta \otimes g_2) + (-1)^p u \backslash\!\!\backslash \partial(\sigma \otimes \eta \otimes g_2).$$

If $u \in C^p(Y; G_1)$ and $\sigma \in C_q(X; G_2)$ then

$$\partial(u \backslash v) = (\delta u) \backslash v + (-1)^p u \backslash \partial v.$$

As a simple corollary we see that the slant product on (co)chains induces a well-defined slant product on (co)homology.

Definition 1.6 (Slant product on (co)homology)**.** The **slant product** on (co)homology is the pairing

$$- \backslash - : H^p(Y; G_1) \otimes H_n(X \times Y; G_2) \to H_{n-p}(X; G_3)$$

where for cocycle $u \in C^p(Y; G_1)$ and cycle $\sigma \in C_n(X \times Y; G_2)$ we set

$$[u] \backslash [\sigma] = [u \backslash \sigma].$$

Lemma 1.7 (Naturality of slant product)**.** *For maps $f : X \to W$ and $g : Y \to Z$ cochain $u \in C^p(Z; G_1)$ and chain $\sigma \in C_n(X \times Y; G_2)$ we have*

$$f_*((g^* u) \backslash \sigma) = u \backslash (f \times g)_* \sigma$$

which induces the corresponding equality on (co)homology groups.

Note this is not quite our formal version of naturality

$$\begin{array}{ccc}
C^p(Y; G_1) \otimes C_n(X \times Y; G_2) & \xrightarrow{-\backslash -} & H_{n-p}(X; G_3) \\
g^* \uparrow \qquad\qquad \downarrow (f \times g)_* & & \downarrow f_* \\
C^p(Z; G_1) \otimes C_n(W \times Z; G_2) & \xrightarrow{-\backslash -} & H_{n-p}(Z; G_3)
\end{array}$$

Definition 1.8 (Cap product)**.** X a space G_1 and G_2 abelian groups with pairing $\omega : G_1 \otimes G_2 \to G_3$. The **cap product** is the pairing

$$- \frown - : C^p(X; G_1) \otimes C_n(X; G_2) \to C_{n-p}(X; G_3)$$

where for $u \in C^p(X; G_1)$, $\sigma \in C_n(X; G_2)$ we set

$$u \frown \sigma = u \backslash \Delta\sigma.$$

where $\Delta : X \to X \times X$ is the diagonal map.

This induces the **cap product** on (co)homology

$$- \frown - : H^p(X; G_1) \otimes H_n(X; G_2) \to H_{n-p}(X; G_3).$$

This formula simplifies to

$$\begin{aligned}
u \frown (\sigma \otimes g_2) &= (\pi_X \circ \Delta\sigma \circ f_{n-p}) \otimes \omega\Big(u(\pi_X \circ \Delta\sigma \circ b_p) \otimes g_2\Big) \\
&= (\sigma \circ f_{n-p}) \otimes \omega\Big(u(\sigma \circ b_p) \otimes g_2\Big)
\end{aligned}$$

Naturality of the slant product gives us naturality for cap product:

Corollary 1.9 (Naturality of cap product)**.** *For map $f : X \to Y$ cochain $u \in C^p(Z; G_1)$ and chain $\sigma \in C_n(X \times Y; G_2)$ we have*

$$f_*((f^*u) \frown \sigma) = u \frown f_*\sigma$$

which induces the corresponding equality on (co)homology groups.

Note this is not quite our formal version of naturality

$$\begin{array}{ccc} C^p(X;G_1) \otimes C_n(X;G_2) & \xrightarrow{-\frown-} & C_{n-p}(X;G_3) \\ f^* \uparrow \quad\quad \downarrow f_* & & \downarrow f_* \\ C^p(Y;G_1) \otimes C_n(Y;G_2) & \xrightarrow{-\frown-} & C_{n-p}(Y;G_3) \end{array}$$

1.3 Cohomology with compact supports

In the case we are most interested where $G_1 = G_2 = R$ with the multiplication pairing we get the formula

$$u \frown r\sigma = \sigma \circ f_{n-q} \otimes u(\sigma \circ b_p) \cdot r$$

Theorem 1.10 (Poincaré Duality)**.** *M an R-orientable n-manifold with fundamental class $[M] \in H_n(M; R)$. Then*

$$- \frown [M] : H^p(X; R) \to H_{n-p}(X; R)$$

is an isomorphism for all p.

Recall that k-cochains of X are functions

$$C^k(X; G) = G^{\mathcal{S}_k(X)}$$

where $\mathcal{S}_k(X)$ is set of singular simplices of X.

Definition 1.11 (Cohomology with compact supports)**.** X a space G abelian. k-cochains with compact support are:

$$C_c^k(X; G) = \left\{ f \in C^k(X; G) \middle| \begin{array}{c} \exists \text{ compact } K \subset X \text{ s.t. } f(\sigma) = 0 \\ \text{for } \sigma \in \mathcal{S}_k(X - K; G) \end{array} \right\}$$

Cohomology with compact supports of X is cohomology $H_c^*(X; G)$ of this cochain complex.