Algebraic Topology II – Lecture 19

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1 Cohomology with compact supports

Recall that k-cochains of X are functions

$$C^k(X;G) = G^{\mathcal{S}_k(X)}$$

where $\mathcal{S}_k(X)$ is set of singular simplices of X.

Definition 1.1 (Cohomology with compact supports)**.** X a space G abelian. k-cochains with compact support are:

$$C_c^k(X;G) = \left\{ f \in C^k(X;G) \middle| \begin{array}{c} \exists \text{ compact } K \subset X \text{ s.t. } f(\sigma) = 0 \\ \text{for } \sigma \in \mathcal{S}_k(X-K;G) \end{array} \right\}$$

Cohomology with compact supports of X is cohomology $H_c^*(X;G)$ of this cochain complex.

Alternatively, recall that

$$C^k(X,A;G) = \left\{ f \in C^k(X;G) \middle| f(\sigma) = 0 \text{ for all } \sigma \in \mathcal{S}(A) \right\}$$

So

$$C_c^k(X;G) = \bigcup_{K \text{ compact}} C^k(X, X-K;G)$$

If we take the directed set $\mathcal{K}$ of compact subsets of X partially ordered by inclusion then the direct system of inclusion maps

$$\{i_{K,L} : (X, X-L) \to (X, X-K)\}_{K,L \in \mathcal{K}}$$

induces a direct system of homomorphisms

$$\{i_{K,L}^* : H^*(X|K;G) \to H^*(X|L;G)\}_{K,L\in\mathcal{K}}$$

where as we did with homology we set $H^k(X|K;G) = H^k(X, X-K;G)$. Taking direct limit

$$\begin{aligned} \varinjlim_{K\in\mathcal{K}} H^*(X|K;G) &= H^*\left(\varinjlim_{K\in\mathcal{K}} C^*(X|K;G) \right) \\ &= H^* C_c^*(X;G) \\ &= H_c^*(X;G) \end{aligned}$$

Lemma 1.2. *If $\mathcal{D}$ is a directed set, $\{h_{D,E} : G_D \to G_E\}_{D,E\in\mathcal{D}}$ is a direct system of group homomorphisms and $\mathcal{S} \subset \mathcal{D}$ is a directed subset such that for all $D \in \mathcal{D}$ there is $S \in \mathcal{S}$ with $D \leq S$ then S is called* ***cofinal*** *and*

$$\varinjlim_{D\in\mathcal{D}} G_D = \varinjlim_{S\in\mathcal{S}} G_S.$$

Proof. Use universal properties to construct homomorphisms from each limit to the other. Then use universal properties twice more to show both compositions must be identity. □

Example 1.3 (Cohomology with compact supports for $\mathbf{R}^n$)*.* For $k \in \mathbf{N}$ let $B_k \subset \mathbf{R}^n$ be closed ball or radius k. Then

$$\begin{aligned} H^k_c(\mathbf{R}^n; G) &= \varinjlim H^k(\mathbf{R}^n|K; G) = \varinjlim H^k(\mathbf{R}^n|B_k; G) \\ &= \varinjlim \widetilde{H}^{k-1}(S^{n-1}; G) = \begin{cases} G, & k = n \\ 0, & k \neq n \end{cases} \end{aligned}$$

We will need relative versions of our various products:

Definition 1.4. A and B subsets of X are an **excisive couple** if $C_*(A) + C_*(B) \subset C_*(A \cup B)$ is a chain homotopy equivalence.

Lemma 1.5. *If A and B are open sets of X or A and B are subcomplexes of the cell complex X then A and B are an excisive couple.*

Notation (Product of topological pairs)

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$$

In what follows all products are defined using representatives in absolute groups and then easily shown to induce well-defined values for relative groups.

Lemma 1.6. *If $X \times B$ and $Y \times A$ is an excisive couple in $X \times Y$ then we get a relative cross product*

$$- \times - : H^p(X, A; G_1) \otimes H^q(Y, B; G_2) \to H^{p+q}((X, A) \times (Y, B); G_3)$$

Lemma 1.7. *If $A, B \subset X$ is an excisive couple then we get a relative cup product*

$$- \smile - : H^p(X, A; G_1) \otimes H^q(X, B; G_2) \to H^{p+q}(X, A \cup B; G_3)$$

Lemma 1.8. *If $A, B \subset X$ is an excisive couple and $X \times B$ and $X \times A$ is an excisive couple in $X \times X$ then*

$$u \smile v = \omega \Delta^*(u \times v)$$

Lemma 1.9. *If $X \times B$ and $Y \times A$ is an excisive couple in $X \times Y$ then we get a relative slant product*

$$-\backslash- : H^p(Y, B; G_1) \otimes H_n((X, A) \times (Y, B); G_2) \to H_{n-p}(X, A; G_3)$$

Lemma 1.10. *If $A, B \subset X$ is an excisive couple then we get a relative cap product*

$$- \frown - : H^p(X, A; G_1) \otimes H_n(X, A \cup B; G_2) \to H_{n-p}(X, B; G_3)$$

Lemma 1.11. *If $A, B \subset X$ is an excisive couple and $X \times B$ and $X \times A$ is an excisive couple in $X \times X$ then*

$$u \frown v = \omega \Delta_*(u \backslash v)$$

Proof of Lemma 1.10

- The chain map

$$r : \tfrac{C_*(X)}{C_*(A)+C_*(B)} \to \tfrac{C_*(X)}{C_*(A \cup B)}$$

- Induces a map on homology

$$r_* : H_*\left(\tfrac{C_*(X)}{C_*(A)+C_*(B)}\right) \to H_*\left(\tfrac{C_*(X)}{C_*(A \cup B)}\right)$$

- Applying the Five Lemma to the long exact sequences of the two short exact sequences

$$0 \to C_*(A) + C_*(B) \to C_*(X) \to \frac{C_*(X)}{C_*(A)+C_*(B)} \to 0$$
$$0 \to C_*(A \cup B) \to C_*(X) \to \frac{C_*(X)}{C_*(A \cup B)} \to 0$$

  we see that r_* is an isomorphism.

- Now given a cocycle $u \in C^p(X, A)$ and a class $\zeta \in H_n(X, A \cup B)$ represent $r_*^{-1}(\zeta)$ as a chain in $z \in C_n(X)$ with boundary $\partial z = a + b \in C_{n-1}(A) + C_{n-1}(B)$

- Then

$$\begin{aligned}\partial(u \frown z) &= \delta u \frown z + (-1)^{n-p} u \frown \partial z \\ &= 0 \frown z + (-1)^{n-p} u \frown (a + b) \\ &= (-1)^{n-p}(af_{n-p-1})u(ab_p) + (-1)^{n-p}(bf_{n-p-1})u(bb_p) \\ &= (-1)^{n-p}(bf_{n-p-1})u(bb_p) \\ &\in C^{n-p-1}(B)\end{aligned}$$

  Hence $u \frown z$ is a cycle in $C^{n-p}(X, B)$.

- If $a \in C_n(A)$ and $b \in C_n(B)$ then

$$\begin{aligned}u \frown (z + a + b) &= (zf_p)u(zb_p) + (af_p)u(ab_p) + (bf_p)u(bb_p) \\ &= (zf_p)u(zb_p) + 0 + (bf_p)u(bb_p)\end{aligned}$$

- So the class $u \frown z$ is well-defined in $\frac{C_{n-p}(X)}{C_{n-p}(B)}$.

$\square$

Lemma 1.12 (Associativity of cross products). *If $u \in H^p(X, A; G_1)$, $v \in H^q(Y, B; G_2)$ and $w \in H^r(Z, C; G_3)$ and all relavant pairs are excisive then we get*

$$u \times (v \times w) = (u \times v) \times w$$

Lemma 1.13 (Associativity of cup products). *If $u \in H^p(X, A; G_1)$, $v \in H^q(X, B; G_2)$ and $w \in H^r(X, C; G_3)$ and all relavant pairs are excisive then we get*

$$u \smile (v \smile w) = (u \smile v) \smile w$$

Lemma 1.14 (Associativity of slant products). *If $u \in H^p(X, A; G_1)$, $v \in H^q(Y, B; G_2)$ and $w \in H^r((X, A) \times (Y, B) \times (Z, C); G_3)$ and all relavant pairs are excisive then we get*

$$(u \times v) \backslash w = u \backslash (v \backslash w)$$

Lemma 1.15 (Associativity of cap products). *If $u \in H^p(X, A; G_1)$, $v \in H^q(X, B; G_2)$ and $w \in H_n(X, A \cup B \cup C; G_3)$ and all relavant pairs are excisive then we get*

$$(u \smile v) \frown w = u \frown (v \frown w)$$

Lemma 1.16 (Existence of units for cross product). *If $1_X \in H^0(X)$ is the class of the augmentation map $\varepsilon : C(X) \to \mathbf{Z}$ for any space X then*

$$\begin{aligned}u \times 1_Y &= \pi_X^*(u) \quad u \in H^p(X, A; G) \\ 1_X \times v &= \pi_Y^*(v) \quad v \in H^q(Y, B; G)\end{aligned}$$

Lemma 1.17 (Existence of units for cup product)**.** *If $1_X \in H^0(X)$ is the class of the augmentation map $\varepsilon : C(X) \to \mathbf{Z}$ for any space X then for all $u \in H^p(X, A; G)$*

$$u \smile 1_X = 1_X \smile u = u$$

Lemma 1.18 (Existence of units for slant product)**.** *For all $v \in H_n(X \times Y, A \times Y; G)$*

$$1_Y \backslash v = \pi_{X*} v$$

Lemma 1.19 (Existence of units for cap product)**.** *For all $u \in H^p(X, B; G)$*

$$1_X \frown v = v$$

- M an R-oriented n-manifold, possibly noncompact.
- For each compact $K \subset M$ we have relative cap product above with $A = M - K$ and $B = \varnothing$

$$- \frown - : H^p(M|K; R) \otimes H_n(M|K; R) \to H_{n-p}(M; R)$$

- Also have $\mu_K \in H_n(M|K; R)$ inducing orientation on points of K.
- Set $D_K : H^p(M|K; R) \to H_{n-p}(M; R)$ to be $D_K(x) = x \frown \mu_K$.
- For compact $K, L \subset X$ with $K \subset L$ the identity map $\mathbf{1}_M : M \to M$ induces the inclusion of topological pairs

$$i : (M, M - L) \to (M, M - K)$$

  where $i_*(\mu_L) = \mu_K$.
- By naturally of cap product for all $x \in H^p(X|K; R)$ we have

$$\begin{aligned} D_K(x) &= x \frown \mu_K \\ &= x \frown i_* \mu_L \\ &= \mathbf{1}_{M*}(i^* x \frown \mu_L) \\ &= i^* x \frown \mu_L \\ &= D_L(i^* x) \end{aligned}$$

- Applying universal property of direct limit we get duality map

$$D_M : H^p_c(M; R) \to H_{n-p}(M; R)$$

Lemma 1.20. *For R-oriented M and open sets $U, V \subset M$ with $M = U \cup V$ and induced orientations we a get commutative diagram of Mayer-Vietoris sequences*

$$\begin{tikzcd} \to H^p_c(U \cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H^p_c(U) \oplus H^p_c(U) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H^p_c(M) \arrow[r] \arrow[d, "D_M"] & H^{p+1}_c(U \cap V) \to \arrow[d, "D_{U\cap V}"] \\ \to H_{n-p}(U \cap V) \arrow[r] & H_{n-p}(U) \oplus H_{n-p}(U) \arrow[r] & H_{n-p}(M) \arrow[r] & H_{n-p-1}(U \cap V) \to \end{tikzcd}$$

Theorem 1.21 (Poincaré Duality for oriented manifolds)**.** *For M an R-oriented n-manifold the duality map*

$$D_M : H^p_c(M; R) \to H_{n-p}(M; R)$$

is an isomorphism for all $p \in \mathbf{Z}$.