Cohomology with compact supports

Recall that $k$-cochains of $X$ are functions

$$C^k(X; G) = G^{S_k(X)}$$

where $S_k(X)$ is set of singular simplices of $X$.

**Definition 1.1** (Cohomology with compact supports). $X$ a space $G$ abelian. $k$-cochains with compact support are:

$$C^k_c(X; G) = \left\{ f \in C^k(X; G) \mid \exists \text{ compact } K \subset X \text{ s.t. } f(\sigma) = 0 \text{ for } \sigma \in S_k(X - K; G) \right\}$$

Cohomology with compact supports of $X$ is cohomology $H^*_{\cdot}(X; G)$ of this cochain complex.

Alternatively, recall that

$$C^k(X, A; G) = \left\{ f \in C^k(X; G) \mid f(\sigma) = 0 \text{ for all } \sigma \in S(A) \right\}$$

So

$$C^k_c(X; G) = \bigcup_{K \text{ compact}} C^k(X, X - K; G)$$

If we take the directed set $\mathcal{K}$ of compact subsets of $X$ partially ordered by inclusion then the direct system of inclusion maps

$$\{i_{K,L}: (X, X - L) \to (X, X - K)\}_{K,L \in \mathcal{K}}$$

induces a direct system of homomorphisms

$$\{i_{K,L}^*: H^*(X|K; G) \to H^*(X|L; G)\}_{K,L \in \mathcal{K}}$$

where as we did with homology we set $H^k(X|K; G) = H^k(X, X - K; G)$. Taking direct limit

$$\lim_{\mathcal{K}} H^*(X|K; G) = H^* \left( \lim_{\mathcal{K}} C^*(X|K; G) \right) = H^*_c(X; G) = H^*_c(X; G)$$

**Lemma 1.2.** If $\mathcal{D}$ is a directed set, $\{h_{D,E}: G_D \to G_E\}_{D,E \in \mathcal{D}}$ is a direct system of group homomorphisms and $S \subset \mathcal{D}$ is a directed subset such that for all $D \in \mathcal{D}$ there is $S \in S$ with $D \leq S$ then $S$ is called **cofinal** and

$$\lim_{D \in \mathcal{D}} G_D = \lim_{S \in \mathcal{S}} G_S.$$
Proof. Use universal properties to construct homomorphisms from each limit to the other. Then use universal properties twice more to show both compositions must be identity.

Example 1.3 (Cohomology with compact supports for $\mathbb{R}^n$). For $k \in \mathbb{N}$ let $B_k \subset \mathbb{R}^n$ be closed ball or radius $k$. Then

$$H^k_\varepsilon(\mathbb{R}^n; G) = \lim\limits_{\to} H^k(\mathbb{R}^n|K; G) = \lim\limits_{\to} H^k(\mathbb{R}^n|B_k; G)$$

$$= \lim\limits_{\to} \tilde{H}^{k-1}(S^{n-1}; G) = \begin{cases} G, & k = n \\ 0, & k \neq n \end{cases}$$

We will need relative versions of our various products:

Definition 1.4. $A$ and $B$ subsets of $X$ are an **excisive couple** if $C_*(A) + C_*(B) \subset C_*(A \cup B)$ is a chain homotopy equivalence.

Lemma 1.5. If $A$ and $B$ are open sets of $X$ or $A$ and $B$ are subcomplexes of the cell complex $X$ then $A$ and $B$ are an excisive couple.

Notation (Product of topological pairs)

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$$

In what follows all products are defined using representatives in absolute groups and then easily shown to induce well-defined values for relative groups.

Lemma 1.6. If $X \times B$ and $Y \times A$ is an excisive couple in $X \times Y$ then we get a relative cross product

$$- \times - : H^p(X, A; G_1) \otimes H^q(Y, B; G_2) \rightarrow H^{p+q}((X, A) \times (Y, B); G_3)$$

Lemma 1.7. If $A, B \subset X$ is an excisive couple then we get a relative cup product

$$- \cup - : H^p(X, A; G_1) \otimes H^q(X, B; G_2) \rightarrow H^{p+q}(X, A \cup B; G_3)$$

Lemma 1.8. If $A, B \subset X$ is an excisive couple and $X \times B$ and $X \times A$ is an excisive couple in $X \times X$ then

$$u \cup v = \omega \Delta^*(u \times v)$$

Lemma 1.9. If $X \times B$ and $Y \times A$ is an excisive couple in $X \times Y$ then we get a relative slant product

$$- \triangle - : H^p(Y, B; G_1) \otimes H_n((X, A) \times (Y, B); G_2) \rightarrow H_{n-p}(X, A; G_3)$$

Lemma 1.10. If $A, B \subset X$ is an excisive couple then we get a relative cap product

$$- \cap - : H^p(X, A; G_1) \otimes H_n(X, A \cup B; G_2) \rightarrow H_{n-p}(X, B; G_3)$$

Lemma 1.11. If $A, B \subset X$ is an excisive couple and $X \times B$ and $X \times A$ is an excisive couple in $X \times X$ then

$$u \cap v = \omega \Delta_*(u \setminus v)$$

Proof of Lemma 1.10

- The chain map

  $$r : \frac{C_*(X)}{C_*(A) + C_*(B)} \rightarrow \frac{C_*(X)}{C_*(A \cup B)}$$

- Induces a map on homology

  $$r_* : H_*(\frac{C_*(X)}{C_*(A) + C_*(B)}) \rightarrow H_*(\frac{C_*(X)}{C_*(A \cup B)})$$
• Applying the Five Lemma to the long exact sequences of the two short exact sequences

\[0 \to C_*(A) + C_*(B) \to C_*(X) \xrightarrow{C_*(X)} \frac{C_*(X)}{C_*(A) + C_*(B)} \to 0\]

\[0 \to C_*(A \cup B) \to C_*(X) \xrightarrow{C_*(X)} \frac{C_*(X)}{C_*(A \cup B)} \to 0\]

we see that \( r_* \) is an isomorphism.

• Now given a cocycle \( u \in C^p(X, A) \) and a class \( \zeta \in H_n(X, A \cup B) \) represent \( r_1^{-1}(\zeta) \) as a chain in \( \zeta \in C_n(X) \) with boundary \( \partial \zeta = a + b \in C_{n-1}(A) + C_{n-1}(B) \)

Then

\[ \partial(u \smile z) = \delta u \smile z + (-1)^{n-p}u \smile \partial z \]

\[ = 0 \smile z + (-1)^{n-p}u \smile (a + b) \]

\[ = (-1)^{n-p}(af_{n-p-1})u(ab_p) + (-1)^{n-p}(bf_{n-p-1})u(bb_p) \]

\[ = (-1)^{n-p}(af_{n-p-1})u(ab_p) + (-1)^{n-p}(bf_{n-p-1})u(bb_p) \]

\[ \in C^{n-p-1}(B) \]

Hence \( u \smile z \) is a cycle in \( C^{n-p}(X, B) \).

• If \( a \in C_n(A) \) and \( b \in C_n(B) \) then

\[ u \smile (z + a + b) = (zf_p)u(zb_p) + (af_p)u(ab_p) + (bf_p)u(bb_p) \]

\[ = (zf_p)u(zb_p) + 0 + (bf_p)u(bb_p) \]

So the class \( u \smile z \) is well-defined in \( \frac{C_{n-p}(X)}{C_{n-p}(B)} \).

\[ \square \]

**Lemma 1.12** (Associativity of cross products). If \( u \in H^p(X, A; G_1) \), \( v \in H^q(Y, B; G_2) \) and \( w \in H^r(Z, C; G_3) \) and all relevant pairs are excisive then we get

\[ u \times (v \times w) = (u \times v) \times w \]

**Lemma 1.13** (Associativity of cup products). If \( u \in H^p(X, A; G_1) \), \( v \in H^q(Y, B; G_2) \) and \( w \in H^r(X, C; G_3) \) and all relevant pairs are excisive then we get

\[ u \smile (v \smile w) = (u \smile v) \smile w \]

**Lemma 1.14** (Associativity of cap products). If \( u \in H^p(X, A; G_1) \), \( v \in H^q(Y, B; G_2) \) and \( w \in H^r((X, A) \times (Y, B) \times (Z, C); G_3) \) and all relevant pairs are excisive then we get

\[ (u \times v) \smile w = u \smile (v \smile w) \]

**Lemma 1.15** (Associativity of cap products). If \( u \in H^p(X, A; G_1) \), \( v \in H^q(Y, B; G_2) \) and \( w \in H_n(X, A \cup B \cup C; G_3) \) and all relevant pairs are excisive then we get

\[ (u \smile v) \smile w = u \smile (v \smile w) \]

**Lemma 1.16** (Existence of units for cross product). If \( 1_X \in H^0(X) \) is the class of the augmentation map \( \varepsilon : C(X) \to \mathbb{Z} \) for any space \( X \) then

\[ u \times 1_Y = \pi^*_u(u) \quad u \in H^p(X, A; G) \]

\[ 1_X \times v = \pi^*_v(v) \quad v \in H^q(Y, B; G) \]
Lemma 1.17 (Existence of units for cup product). If \(1_X \in H^0(X)\) is the class of the augmentation map \(\varepsilon : C(X) \to \mathbb{Z}\) for any space \(X\) then for all \(u \in H^p(X,A;G)\)

\[
u \circ 1_X = 1_X \circ u = u
\]

Lemma 1.18 (Existence of units for slant product). For all \(v \in H_n(X \times Y,A \times Y;G)\)

\[
1_Y \\downarrow v = \pi_X^* v
\]

Lemma 1.19 (Existence of units for cap product). For all \(u \in H^p(X,B;G)\)

\[
1_X \cap v = v
\]

- \(M\) an \(R\)-oriented \(n\)-manifold, possibly noncompact.
- For each compact \(K \subseteq M\) we have relative cap product above with \(A = M - K\) and \(B = \emptyset\)

\[
- \cap - : H^p(M|K;R) \otimes H_n(M|K;R) \to H_{n-p}(M;R)
\]
- Also have \(\mu_K \in H_n(M|K;R)\) inducing orientation on points of \(K\).
- Set \(D_K : H^p(M|K;R) \to H_{n-p}(M;R)\) to be \(D_K(x) = x \cap \mu_K\).
- For compact \(K,L \subseteq X\) with \(K \subseteq L\) the identity map \(1_M : M \to M\) induces the inclusion of topological pairs

\[
i : (M,M - L) \to (M,M - K)
\]

where \(i_*(\mu_L) = \mu_K\).
- By naturally of cap product for all \(x \in H^p(X|K;R)\) we have

\[
D_K(x) = x \cap \mu_K
\]

\[
= x \cap i_* \mu_L
\]

\[
= 1_M(i^* x \cap \mu_L)
\]

\[
= i^* x \cap \mu_L
\]

\[
= D_L(i^* x)
\]

- Applying universal property of direct limit we get duality map

\[
D_M : H^p_c(M;R) \to H_{n-p}(M;R)
\]

Lemma 1.20. For \(R\)-oriented \(M\) and open sets \(U,V \subseteq M\) with \(M = U \cup V\) and induced orientations we a get commutative diagram of Mayer-Vietoris sequences

\[
\begin{array}{ccccccccc}
& & H_p(U \cap V) & \longrightarrow & H_p(U) \oplus H_p(U) & \longrightarrow & H_p(M) & \longrightarrow & H_{p+1}(U \cap V) & \longrightarrow \\
& & D_{U \cap V} & & D_U \oplus D_V & & D_M & & D_{U \cap V} & \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& H_{n-p}(U \cap V) & \to & H_{n-p}(U) \oplus H_{n-p}(U) & \to & H_{n-p}(M) & \to & H_{n-p-1}(U \cap V) & \to
\end{array}
\]

Theorem 1.21 (Poincaré Duality for oriented manifolds). For \(M\) an \(R\)-oriented \(n\)-manifold the duality map

\[
D_M : H^p_c(M;R) \to H_{n-p}(M;R)
\]

is an isomorphism for all \(p \in \mathbb{Z}\).