Algebraic Topology II – Lecture 26

Nathan Broaddus

March 25, 2015

1 Hopf algebras

1.1 H-spaces

Definition 1.1 (H-space)**.** An **H-space** (X, μ, e) is

1. a topological space X
2. with continuous "multiplication" map $\mu : X \times X \to X$
3. and identity element $e \in X$

such that the maps $\ell : X \to X$ and $r : X \to X$ with $r(x) = \mu(x, e)$ and $\ell(x) = \mu(e, x)$ are homotopic to $\mathbf{1}_X : X \to X$ via homotopies fixing e.

Example 1.2 (H-spaces).
Let $D \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$

1. $\mathrm{Mat}_n(D)$
   - Topologize the set $\mathrm{Mat}_n(D)$ of $n \times n$ matrices over D as $D^{(n^2)}$
   - Multiplication is continuous since polynomials in entries are continuous
2. $\mathrm{GL}_n(D)$
   - $\mathrm{GL}_n(D)$ is subspace of $\mathrm{Mat}_n(D)$ of invertible elements.
   - $\mathrm{GL}_n(\mathbf{R})$ and $\mathrm{GL}_n(\mathbf{C})$ characterized by evidently open condition that determinant is nonzero.
   - Can embed $\mathrm{Mat}_n(\mathbf{H})$ in $\mathrm{Mat}_{2n}(\mathbf{C})$ then $\mathrm{GL}_n(\mathbf{H})$ is matrices with nonzero determinant (taken in $\mathrm{Mat}_{2n}(\mathbf{C})$).
3. $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}(n)$
   - $\mathrm{O}(n)$ is compact subspace of $\mathrm{GL}_n(\mathbf{R})$ with $AA^T = I$ where A^T is transpose of A.
   - $\mathrm{U}(n)$ is compact subspace of $\mathrm{GL}_n(\mathbf{C})$ with with $AA^* = I$ where A^* is $\overline{A^T}$.
   - $\mathrm{Sp}(n)$ is compact subspace of $\mathrm{GL}_n(\mathbf{H})$ with $AA^* = I$ (where $A \in \mathrm{GL}_{2n}(\mathbf{C})$).
4. In particular S^0, S^1, S^3 and S^7 are H-spaces since
   - $S^0 = \mathrm{O}(1)$
   - $S^1 = \mathrm{U}(1)$
   - $S^3 = \mathrm{Sp}(1)$
   - $S^7 \subset \mathbf{O}$ octonions with modulus 1

5. Polynomial algebras $\mathbf{R}[x]$ and $\mathbf{C}[x]$
   - Topologize as $\bigcup_{n\in\mathbf{N}} \mathbf{R}^n$ and $\bigcup_{n\in\mathbf{N}} \mathbf{C}^n$ with coproduct topology (U is open if $U \cap \mathbf{R}^n$ is open for each n).
6. $\mathbf{R}P^\infty$ and $\mathbf{C}P^\infty$
   - $$\mathbf{R}P^\infty = \mathbf{R}[x] - \{0\} \big/ \mathbf{R}^*$$
     where $\mathbf{R}^*$ acts on $\mathbf{R}[x]$ by multiplication
   - $$\mathbf{C}P^\infty = \mathbf{C}[x] - \{0\} \big/ \mathbf{C}^*$$
     where $\mathbf{C}^*$ acts on $\mathbf{C}[x]$ by multiplication
   - Multiplication in $\mathbf{R}[x]$ and $\mathbf{C}[x]$ gives well-defined multiplication in $\mathbf{R}P^\infty$ and $\mathbf{C}P^\infty$
7. Lens space $L^\infty(q)$
   - $$L^\infty(q) = \mathbf{C}[x] - \{0\} \big/ \mathbf{R}^{>0} \times \mathbf{Z}$$
     where $(r, n) \in \mathbf{R}^{>0} \times \mathbf{Z}$ acts on $\mathbf{C}[x]$ by multiplication by $re^{2\pi i n/q}$
   - Note $L^\infty(2) \cong \mathbf{R}P^\infty$ as H-spaces.
8. James reduced product $J(X)$
   - X a space with base point $e \in X$
   - The **James reduced product** is
     $$J(X) = \coprod_{n\in\mathbf{N}} X^n \Big/ \sim$$
     where $\sim$ is equivalence relation generated by
     $$(x_1, \cdots, x_i, e, x_{i+1}, \cdots, x_n) \sim (x_1, \cdots, x_n)$$
   - Multiplication in $J(X)$ is
     $$(x_1, \cdots, x_n) \cdot (y_1, \cdots, y_m) = (x_1, \cdots, x_n, y_1, \cdots, y_m)$$
   - Note: If X is a cell complex then $J(X)$ has cell complex structure.
9. Symmetric product $SP(X)$
   - $$SP(X) = J(X) \big/ \mathrm{Sym}_\infty$$
     where $\mathrm{Sym}_\infty = \bigcup_{n\in\mathbf{N}} \mathrm{Sym}_n$ is the infinite symmetric group acting on $J(X)$ by permuting coordinates.