1 Hopf algebras

1.1 Hopf algebras

Definition 1.1 (Tensor product of graded algebras). If $A = \bigoplus_{n \in \mathbb{N}} A^n$ and $B = \bigoplus_{n \in \mathbb{N}} B^n$ are graded $R$-algebras then $A \otimes B$ is a graded $R$-algebra with grading

$$(A \otimes B)_n = \bigoplus_{p=0}^{n} A^p \otimes B^{n-p}$$

and product

$$(\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta_2) = (-1)^{|\beta_1||\alpha_2|} (\alpha_1 \alpha_2) \otimes (\beta_1 \beta_2)$$

Notice that $A$ and $B$ are commutative if and only if $A \otimes B$ is.

Definition 1.2 (Hopf Algebra). A Hopf algebra over the commutative ring $R$ is a graded $R$-algebra (not necessarily commutative or associative)

$$A = \bigoplus_{n \in \mathbb{N}} A^n$$

with an element $1 \in A^0$ such that $r \mapsto r \cdot 1$ gives an isomorphism $R \rightarrow A^0$ (we say $A$ is connected). Further we have a graded $R$-algebra homomorphism

$$\Delta : A \rightarrow A \otimes A$$

called the coproduct such that for all $\alpha \in A^n$ with $n > 0$

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_{i=1}^{m} a'_i \otimes b'_i$$

where $|a'_i| > 0$ and $|b'_i| > 0$.

Suppose $(X, \mu, e)$ is an H-space. Then

$$\mu : X \times X \rightarrow X$$

induces

$$\mu^* : H^*(X) \rightarrow H^*(X \times X)$$

If $H^*(X; R)$ is free then by the Künneth formula for cohomology the cross product gives an isomorphism

$$\rho : H^*(X) \otimes H^*(X) \rightarrow H^*(X \times X)$$

where $\rho(u \otimes v) = u \times v$
Proposition 1.3. If $X$ is a path connected $H$-space such that $H^n(X)$ is a finitely generated free $R$-module for each $n$ then $H^*(X)$ is a commutative, associative Hopf algebra where

$$\Delta = \rho^{-1}\mu^*$$

Before we prove Proposition 1.3 we point out a fundamental property of the cross product on cohomology that we previously omitted.

Lemma 1.4 (Naturality of cohomology cross product). For maps $f : X \to W$ and $g : Y \to Z$ and cochains $u \in H^p(W)$ and $v \in H^q(Z)$ we have

$$(f \times g)^*(u \times v) = (f^* u) \times (g^* v)$$

which induces the corresponding equality on cohomology groups. That is, we get commutativity of

$$H^p(W) \otimes H^q(Z) \xrightarrow{(f \times g)^*} H^{p+q}(W \times Z)$$

$$\xrightarrow{f^* \otimes g^*} H^p(X) \otimes H^q(Y) \xrightarrow{(f \times g)^*} H^{p+q}(X \times Y)$$

Proof of Proposition 1.3

Claim 1: $H^0(X)$ is connected.

- The class $1 = [e]$ of the the augmentation homomorphism $\varepsilon : C_0(X) \to R$ is a unit in $H^0(X)$
- $X$ is path connected so the homomorphism $r \mapsto r \cdot 1$ is an isomorphism $R \to H^0(X)$.
- Thus $H^*(X)$ is connected.

Claim 2: $\Delta$ is a coproduct

- We have the inclusions:

$$i : \{e\} \to X$$
$$1_X : X \to X$$

and the homeomorphism

$$j : X \to X \times \{e\}$$

- By the H-space axiom we have $\mu \circ (1_X \times i) \circ j \simeq 1_X$
- Set the “projection” $P : H^*(X) \otimes H^*(X) \to H^*(X)$

$$P = [(1_X \times i) \circ j]^* \rho = j^*(1_X \times i)^* \rho$$

where as above $\rho(u \otimes v) = u \times v$.
- By naturality of cross product we have $P = j^* \rho(1_X \otimes i^*)$
• In summary, we have commutative

\[
\begin{array}{c}
\xymatrix{ H^*(X) \otimes H^*(X) \ar[r]^\rho \ar[d]^{1_X \otimes i^*} & H^*(X \times X) \\
H^*(X) \otimes H^*(e) \ar[r]^\rho \ar[d]^{(1_X \times i)^*} & H^*(X \times \{e\}) \\
\downarrow P \ar[r]^{j^*} & H^*(X) 
}\end{array}
\]

• If \( \alpha \in H^*(X) \) and \( \beta \in H^*(X) \) then

\[
P(\alpha \otimes \beta) = j^* \rho(1_X^* \otimes i^*)(\alpha \otimes \beta)
= j^* \rho(1_X^* \alpha \otimes i^* \beta)
= j^*(1_X^* \alpha \times i^* \beta)
\]

• Notice that \( i : \{e\} \to X \) induces an isomorphism \( i^* : H^0(X) \to H^0(e) \) so

• And induces the zero map \( i^* : H^n(X) \to H^n(e) \) for \( n > 0 \).

• So for \( 1 = [e] \in H^0(X) \) we get

\[
P(\alpha \otimes 1) = j^*(1_X^* \alpha \times i^* 1)
= j^*(\alpha \times 1)
= j^*(\pi_X^* \alpha)
= (\pi_X \circ j)^* \alpha
= 1_X^* \alpha
= \alpha
\]

• and for \( \beta \in H^n(X) \) with \( n > 0 \) we get

\[
P(\alpha \otimes \beta) = j^*(1_X^* \alpha \times i^* \beta)
= j^*(\alpha \times 0)
= 0
\]

• Given arbitrary element \( \sum_{i=1}^m \alpha_i \otimes \beta_i \in H^*(X) \otimes H^*(X) \) moving scalars across tensor product we may assume that if \( |\beta_i| = 0 \) then \( \beta_i = 1 \).

• By linearity we conclude that

\[
P \left( \sum_{i=1}^m \alpha_i \otimes \beta_i \right) = \sum_{|\beta_i| = 0} \alpha_i
\]

• Now given \( \alpha \in H^n(X) \) consider the element \( \Delta(\alpha) \in H^n(X) \otimes H^n(X) \)

\[
P(\Delta \alpha) = j^*(1_X^* \times i^*) \rho \rho^{-1} \mu^* \alpha
= j^*(1_X^* \times i^*) \mu^* \alpha
= (\mu \circ (1_X \times i) \circ j)^* \alpha
= 1_X^* \alpha
= \alpha
\]
• Thus \( \Delta(\alpha) = a_1 \otimes 1 + \cdots + a_k \otimes 1 + \sum_{i=1}^{m} a_i' \otimes b_i' \) with \(|b_i'| > 0\) and \(\sum_{i=1}^{k} a_k = \alpha\)

• Hence \( \Delta(\alpha) = \alpha \otimes 1 + \sum_{i=1}^{m} a_i' \otimes b_i' \) with \(|b_i'| > 0\).

• Mirror argument gives projection

\[
Q : H^*(X) \otimes H^*(X) \to H^*(X)
\]

such that \(Q(1 \otimes \alpha) = \alpha\) and \(Q(\beta \otimes \alpha) = 0\) if \(|\beta| > 0\).

• Applying \(Q\) to \(\Delta(\alpha) = \alpha \otimes 1 + \sum_{i=1}^{m} a_i' \otimes b_i'\) we see that

\[
\sum_{i=1}^{m} a_i' \otimes b_i' = 1 \otimes \alpha + \sum_{i=1}^{k} a_i'' \otimes b_i''
\]

with \(|a_i''| > 0\) and \(|b_i''| > 0\).

• Thus for \(\alpha \in H^n(X)\) with \(n > 0\) we have

\[
\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_{i=1}^{m} a_i'' \otimes b_i''
\]

where \(|a_i''| > 0\) and \(|b_i''| > 0\).

\[\square\]

\textit{Example 1.5 (Polynomial ring as a Hopf algebra).}

• What Hopf algebra structures can we put on the polynomial ring \(R[\alpha]\) (also known as free associative \(R\)-algebra with generator \(\alpha\))?  

• Must define coproduct \(\Delta : R[\alpha] \to R[\alpha] \otimes R[\alpha]\).

• \(\Delta\) is a graded \(R\)-algebra homomorphism so \(\Delta\) is determined by \(\Delta(\alpha)\)

• By Hopf algebra property

\[
\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_{i=1}^{m} a_i'' \otimes b_i''
\]

where \(|a_i''| > 0\) and \(|b_i''| > 0\).

• Since \(\Delta\) preserves grading we must have \(|a_i''| + |b_i''| = |\alpha|\).

• But \(a_i'' \in R[\alpha]\) and \(|a_i''| > 0\) so \(|a_i''| \geq |\alpha|\) (similarly \(|b_i''| \geq |\alpha|\)).

• Hence \(|a_i''| + |b_i''| \geq 2|\alpha| > |\alpha|\).

• Thus \textbf{unique} Hopf algebra structure on \(R[\alpha]\) has coproduct

\[
\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha.
\]

• Case I: Suppose \(|\alpha|\) is even

  – Then since \(\Delta\) is an \(R\)-algebra homomorphism we must have

\[
\Delta(\alpha^n) = \Delta(\alpha)^n
= (\alpha \otimes 1 + 1 \otimes \alpha)^n
= \sum_{i=0}^{n} \binom{n}{i} \alpha^i \otimes \alpha^{n-i}
\]

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Case II: Suppose $|\alpha|$ is odd

- Note that in this case $R[\alpha]$ is not a commutative algebra unless $2 = 0$ in $R$ since commutativity would require $\alpha^2 = -\alpha^2$ and hence $2\alpha^2 = 0$.
- Then
  \[
  \Delta(\alpha^2) = \Delta(\alpha)^2 \\
  = (\alpha \otimes 1 + 1 \otimes \alpha)(\alpha \otimes 1 + 1 \otimes \alpha) \\
  = \alpha^2 \otimes 1 + \alpha \otimes \alpha - \alpha \otimes \alpha + 1 \otimes \alpha^2 \\
  = \alpha^2 \otimes 1 + 1 \otimes \alpha^2
  \]

- Hence $\alpha^2$ has even degree and by Case I we get
  \[
  \Delta(\alpha^{2n}) = \Delta(\alpha^2)^n \\
  = \sum_{i=0}^{n} \binom{n}{i} \alpha^{2i} \otimes \alpha^{2n-2i}
  \]

- and
  \[
  \Delta(\alpha^{2n+1}) = \Delta(\alpha)\Delta(\alpha^{2n}) \\
  = (\alpha \otimes 1 + 1 \otimes \alpha) \sum_{i=0}^{n} \binom{n}{i} \alpha^{2i} \otimes \alpha^{2n-2i} \\
  = \sum_{i=0}^{n} \binom{n}{i} \alpha^{2i+1} \otimes \alpha^{2n-2i} + \sum_{i=0}^{n} \binom{n}{i} \alpha^{2i} \otimes \alpha^{2n-2i+1} \\
  = \sum_{i=0}^{n} \binom{n}{i} (\alpha^{2i} \otimes \alpha^{2n-2i+1} + \alpha^{2i+1} \otimes \alpha^{2n-2i})
  \]

**Example 1.6 (Exterior algebra $\Lambda_R[\alpha]$ as a Hopf algebra).**

- The exterior algebra $\Lambda_R[\alpha]$ is the quotient of the free associative $R$-algebra $R[\alpha]$ by the homogeneous relation $\alpha^2 = 0$.
- Thus as an $R$-module we have $\Lambda_R[\alpha] \cong R \oplus R\alpha$
- Since $\Lambda_R[\alpha]$ has no elements with degree other than 0 and $|\alpha|$ the only possible coproduct on $\Lambda_R[\alpha]$ is $\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$

**Claim 1: If $|\alpha|$ is odd then $\Delta$ is a coproduct**

- The free associative $R$-algebra $R[\alpha]$ has a graded module homomorphism $\Delta' : R[\alpha] \to \Lambda_R[\alpha] \otimes \Lambda_R[\alpha]$ with $\Delta'(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$
- It induces $\Delta : \Lambda_R[\alpha] \to \Lambda_R[\alpha] \otimes \Lambda_R[\alpha]$ if $\Delta'(\alpha^2) = 0$.

  \[
  \Delta'(\alpha^2) = \Delta'(\alpha)^2 \\
  = (\alpha \otimes 1 + 1 \otimes \alpha)(\alpha \otimes 1 + 1 \otimes \alpha) \\
  = \alpha^2 \otimes 1 + \alpha \otimes \alpha - \alpha \otimes \alpha + 1 \otimes \alpha^2 \\
  = 0
  \]

**Claim 2: If $|\alpha|$ is even then $\Delta$ is a coproduct if and only if $2 = 0$ in $R$**
• Again we have $R$-algebra homomorphism $\Delta' : R[\alpha] \to \Lambda_R[\alpha] \otimes \Lambda_R[\alpha]$ with $\Delta'(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$

• It induces $\Delta : \Lambda_R[\alpha] \to \Lambda_R[\alpha] \otimes \Lambda_R[\alpha]$ if $\Delta'(\alpha^2) = 0$.

\[
\Delta'(\alpha^2) = \Delta'(\alpha)^2 \\
= (\alpha \otimes 1 + 1 \otimes \alpha)(\alpha \otimes 1 + 1 \otimes \alpha) \\
= \alpha^2 \otimes 1 + \alpha \otimes \alpha + \alpha \otimes \alpha + 1 \otimes \alpha^2 \\
= 2\alpha \otimes \alpha
\]

which is 0 if and only if $2 = 0$ in $R$. 

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