Algebraic Topology II – Lecture 27

Nathan Broaddus

March 27, 2015

1 Hopf algebras

1.1 Hopf algebras

Definition 1.1 (Tensor product of graded algebras)**.** If $A = \bigoplus_{n\in\mathbf{N}} A^n$ and $B = \bigoplus_{n\in\mathbf{N}} B^n$ are graded R-algebras then $A \otimes B$ is a graded R-algebra with grading

$$(A \otimes B)_n = \bigoplus_{p=0}^{n} A^p \otimes B^{n-p}$$

and product

$$(\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta_2) = (-1)^{|\beta_1||\alpha_2|}(\alpha_1\alpha_2) \otimes (\beta_1\beta_2)$$

Notice that A and B are commutative if and only if $A \otimes B$ is.

Definition 1.2 (Hopf Algebra)**.** A **Hopf algebra** over the commutative ring R is a graded R-algebra (not necessarily commutative or associative)

$$A = \bigoplus_{n\in\mathbf{N}} A^n$$

with an element $1 \in A^0$ such that $r \mapsto r \cdot 1$ gives an isomorphism $R \to A^0$ (we say A is **connected**). Further we have a graded R-algebra homomorphism

$$\Delta : A \to A \otimes A$$

called the **coproduct** such that for all $\alpha \in A^n$ with $n > 0$

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_{i=1}^{m} a'_i \otimes b'_i$$

where $|a'_i| > 0$ and $|b'_i| > 0$.

Suppose (X, μ, e) is an H-space. Then

$$\mu : X \times X \to X$$

induces

$$\mu^* : H^*(X) \to H^*(X \times X)$$

If $H^*(X; R)$ is free then by the Künneth formula for cohomology the cross product gives an isomorphism

$$\rho : H^*(X) \otimes H^*(X) \to H^*(X \times X)$$

where $\rho(u \otimes v) = u \times v$

Proposition 1.3. *If X is a path connected H-space such that $H^n(X)$ is a finitely generated free R-module for each n then $H^*(X)$ is a commutative, associative Hopf algebra where*

$$\Delta = \rho^{-1}\mu^*$$

Before we prove Proposition 1.3 we point out a fundamental property of the cross product on cohomology that we previously omitted.

Lemma 1.4 (Naturality of cohomology cross product)**.** *For maps $f : X \to W$ and $g : Y \to Z$ and cochains $u \in H^p(W)$ and $v \in H^q(Z)$ we have*

$$(f \times g)^*(u \times v) = (f^*u) \times (g^*v)$$

which induces the corresponding equality on cohomology groups. That is, we get commutativity of

$$\begin{tikzcd} H^p(W) \otimes H^q(Z) \arrow[r, "- \times -"] \arrow[d, "f^* \otimes g^*"] & H^{p+q}(W \times Z) \arrow[d, "(f \times g)^*"] \\ H^p(X) \otimes H^q(Y) \arrow[r, "- \times -"] & H^{p+q}(X \times Y) \end{tikzcd}$$

Proof of Proposition 1.3

Claim 1: $H^0(X)$ *is connected.*

- The class $1 = [\varepsilon]$ of the the augmentation homomorphism $\varepsilon : C_0(X) \to R$ is a unit in $H^0(X)$
- X is path connected so the homomorphism $r \mapsto r \cdot 1$ is an isomorphism $R \to H^0(X)$.
- Thus $H^*(X)$ is connected.

Claim 2: Δ *is a coproduct*

- We have the inclusions:

$$i : \{e\} \to X$$
$$\mathbf{1}_X : X \to X$$

  and the homeomorphism

$$j : X \to X \times \{e\}$$

- By the H-space axiom we have $\mu \circ (\mathbf{1}_X \times i) \circ j \simeq \mathbf{1}_X$
- Set the "projection"

$$P : H^*(X) \otimes H^*(X) \to H^*(X)$$

  to be $P = [(\mathbf{1}_X \times i) \circ j]^*\rho = j^*(\mathbf{1}_X \times i)^*\rho$ where as above $\rho(u \otimes v) = u \times v$.
- By naturality of cross product we have

$$P = j^*\rho(\mathbf{1}_X^* \otimes i^*)$$

- In summary, we have commutative

$$\begin{tikzcd}
H^*(X)\otimes H^*(X) \arrow[r, "\rho"] \arrow[d, "\mathbf{1}_X^* \otimes i^*"] \arrow[ddr, bend right, "P"'] & H^*(X\times X) \arrow[d, "(\mathbf{1}_X \times i)^*"] \\
H^*(X)\otimes H^*(e) \arrow[r, "\rho"] & H^*(X\times \{e\}) \arrow[d, "j^*"] \\
& H^*(X)
\end{tikzcd}$$

- If $\alpha \in H^*(X)$ and $\beta \in H^*(X)$ then

$$\begin{aligned} P(\alpha\otimes\beta) &= j^*\rho(\mathbf{1}_X^* \otimes i^*)(\alpha\otimes\beta) \\ &= j^*\rho(\mathbf{1}_X^*\,\alpha \otimes i^*\beta) \\ &= j^*(\mathbf{1}_X^*\,\alpha \times i^*\beta) \end{aligned}$$

- Notice that $i : \{e\} \to X$ induces an isomorphism $i^* : H^0(X) \to H^0(e)$ so
- And induces the zero map $i^* : H^n(X) \to H^n(e)$ for $n > 0$.
- So for $1 = [\varepsilon] \in H^0(X)$ we get

$$\begin{aligned} P(\alpha\otimes 1) &= j^*(\mathbf{1}_X^*\,\alpha \times i^*1) \\ &= j^*(\alpha\times 1) \\ &= j^*(\pi_X^*\alpha) \\ &= (\pi_X \circ j)^*\alpha \\ &= \mathbf{1}_X^*\,\alpha \\ &= \alpha \end{aligned}$$

- and for $\beta \in H^n(X)$ with $n > 0$ we get

$$\begin{aligned} P(\alpha\otimes\beta) &= j^*(\mathbf{1}_X^*\,\alpha \times i^*\beta) \\ &= j^*(\alpha\times 0) \\ &= 0 \end{aligned}$$

- Given arbitrary element $\sum_{i=1}^m \alpha_i\otimes\beta_i \in H^*(X)\otimes H^*(X)$ moving scalars across tensor product we may assume that if $|\beta_i| = 0$ then $\beta_i = 1$.
- By linearity we conclude that

$$P\left(\sum_{i=1}^m \alpha_i\otimes\beta_i\right) = \sum_{|\beta_i|=0} \alpha_i$$

- Now given $\alpha \in H^n(X)$ consider the element $\Delta(\alpha) \in H^*(X)\otimes H^*(X)$

$$\begin{aligned} P(\Delta\alpha) &= j^*(\mathbf{1}_X^* \times i^*)\rho\rho^{-1}\mu^*\alpha \\ &= j^*(\mathbf{1}_X^* \times i^*)\mu^*\alpha \\ &= (\mu\circ(\mathbf{1}_X \times i)\circ j)^*\alpha \\ &= \mathbf{1}_X^*\,\alpha \\ &= \alpha \end{aligned}$$

- Thus $\Delta(\alpha) = a_1 \otimes 1 + \cdots + a_k \otimes 1 + \sum_{i=1}^m a_i' \otimes b_i'$ with $|b_i| > 0$ and $\sum_{i=1}^k a_k = \alpha$
- Hence $\Delta(\alpha) = \alpha \otimes 1 + \sum_{i=1}^m a_i' \otimes b_i'$ with $|b_i'| > 0$.
- Mirror argument gives projection

$$Q : H^*(X) \otimes H^*(X) \to H^*(X)$$

  such that $Q(1 \otimes \alpha) = \alpha$ and $Q(\beta \otimes \alpha) = 0$ if $|\beta| > 0$
- Applying Q to $\Delta(\alpha) = \alpha \otimes 1 + \sum_{i=1}^m a_i' \otimes b_i'$ we see that

$$\sum_{i=1}^m a_i' \otimes b_i' = 1 \otimes \alpha + \sum_{i=1}^k a_i'' \otimes b_i''$$

  with $|a_i''| > 0$ and $|b_i''| > 0$.
- Thus for $\alpha \in H^n(X)$ with $n > 0$ we have

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_{i=1}^m a_i'' \otimes b_i''$$

  where $|a_i''| > 0$ and $|b_i''| > 0$.

□

Example 1.5 (Polynomial ring as a Hopf algebra).

- What Hopf algebra structures can we put on the polynomial ring $R[\alpha]$ (also known as free associative R-algebra with generator α)?
- Must define coproduct $\Delta : R[\alpha] \to R[\alpha] \otimes R[\alpha]$.
- Δ is a graded R-algebra homomorphism so Δ is determined by $\Delta(\alpha)$
- By Hopf algebra property

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_{i=1}^m a_i'' \otimes b_i''$$

  where $|a_i''| > 0$ and $|b_i''| > 0$.
- Since Δ preserves grading we must have $|a_i''| + |b_i''| = |\alpha|$.
- But $a_i'' \in R[\alpha]$ and $|a_i''| > 0$ so $|a_i''| \geq |\alpha|$ (similarly $|b_i''| \geq |\alpha|$).
- Hence $|a_i''| + |b_i''| \geq 2|\alpha| > |\alpha|$.
- Thus **unique** Hopf algebra structure on $R[\alpha]$ has coproduct

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha.$$

- Case I: Suppose $|\alpha|$ is even
  - Then since Δ is an R-algebra homomorphism we must have

$$\begin{aligned} \Delta(\alpha^n) &= \Delta(\alpha)^n \\ &= (\alpha \otimes 1 + 1 \otimes \alpha)^n \\ &= \sum_{i=0}^n \binom{n}{i} \alpha^i \otimes \alpha^{n-i} \end{aligned}$$

- Case II: Suppose $|\alpha|$ is odd
  - Note that in this case $R[\alpha]$ is **not a commutative algebra** unless $2 = 0$ in R since commutativity would require $\alpha^2 = -\alpha^2$ and hence $2\alpha^2 = 0$.
  - Then

$$\begin{aligned}
\Delta(\alpha^2) &= \Delta(\alpha)^2 \\
&= (\alpha \otimes 1 + 1 \otimes \alpha)(\alpha \otimes 1 + 1 \otimes \alpha) \\
&= \alpha^2 \otimes 1 + \alpha \otimes \alpha - \alpha \otimes \alpha + 1 \otimes \alpha^2 \\
&= \alpha^2 \otimes 1 + 1 \otimes \alpha^2
\end{aligned}$$

  - Hence α^2 has even degree and by Case I we get

$$\begin{aligned}
\Delta(\alpha^{2n}) &= \Delta(\alpha^2)^n \\
&= \sum_{i=0}^{n} \binom{n}{i} \alpha^{2i} \otimes \alpha^{2n-2i}
\end{aligned}$$

  - and

$$\begin{aligned}
\Delta(\alpha^{2n+1}) &= \Delta(\alpha)\Delta(\alpha^{2n}) \\
&= (\alpha \otimes 1 + 1 \otimes \alpha) \sum_{i=0}^{n} \binom{n}{i} \alpha^{2i} \otimes \alpha^{2n-2i} \\
&= \sum_{i=0}^{n} \binom{n}{i} \alpha^{2i+1} \otimes \alpha^{2n-2i} + \sum_{i=0}^{n} \binom{n}{i} \alpha^{2i} \otimes \alpha^{2n-2i+1} \\
&= \sum_{i=0}^{n} \binom{n}{i} \left(\alpha^{2i} \otimes \alpha^{2n-2i+1} + \alpha^{2i+1} \otimes \alpha^{2n-2i}\right)
\end{aligned}$$

Example 1.6 (Exterior algebra $\Lambda_R[\alpha]$ as a Hopf algebra).

- The exterior algebra $\Lambda_R[\alpha]$ is the quotient of the free associative R-algebra $R[\alpha]$ by the homogeneous relation $\alpha^2 = 0$.
- Thus as an R-module we have $\Lambda_R[\alpha] \cong R \oplus R\alpha$
- Since $\Lambda_R[\alpha]$ has no elements with degree other than 0 and $|\alpha|$ the only possible coproduct on $\Lambda_R[\alpha]$ is $\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$

Claim 1: *If $|\alpha|$ is odd then Δ is a coproduct*

- The free associative R-algebra $R[\alpha]$ has a graded module homomorphism $\Delta' : R[\alpha] \to \Lambda_R[\alpha] \otimes \Lambda_R[\alpha]$ with $\Delta'(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$
- It induces $\Delta : \Lambda_R[\alpha] \to \Lambda_R[\alpha] \otimes \Lambda_R[\alpha]$ if $\Delta'(\alpha^2) = 0$.

$$\begin{aligned}
\Delta'(\alpha^2) &= \Delta'(\alpha)^2 \\
&= (\alpha \otimes 1 + 1 \otimes \alpha)(\alpha \otimes 1 + 1 \otimes \alpha) \\
&= \alpha^2 \otimes 1 + \alpha \otimes \alpha - \alpha \otimes \alpha + 1 \otimes \alpha^2 \\
&= 0
\end{aligned}$$

Claim 2: *If $|\alpha|$ is even then Δ is a coproduct if and only if $2 = 0$ in R*

- Again we have R-algebra homomorphism $\Delta' : R[\alpha] \to \Lambda_R[\alpha] \otimes \Lambda_R[\alpha]$ with $\Delta'(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$
- It induces $\Delta : \Lambda_R[\alpha] \to \Lambda_R[\alpha] \otimes \Lambda_R[\alpha]$ if $\Delta'(\alpha^2) = 0$.

$$\begin{aligned}
\Delta'(\alpha^2) &= \Delta'(\alpha)^2 \\
&= (\alpha \otimes 1 + 1 \otimes \alpha)(\alpha \otimes 1 + 1 \otimes \alpha) \\
&= \alpha^2 \otimes 1 + \alpha \otimes \alpha + \alpha \otimes \alpha + 1 \otimes \alpha^2 \\
&= 2\alpha \otimes \alpha
\end{aligned}$$

which is 0 if and only if $2 = 0$ in R.