Homology theory

Lecture 10 - 1/24/2011
Axiom 4: Additivity
Computations

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Cellular Homology

Math 757
Homology theory

January 27, 2011
Axiom 4: Additivity

**Definition 90 (Direct sum in Ab)**

Given abelian groups \( \{ A_\alpha \} \), the **direct sum** is an abelian group \( A \) with homomorphisms

\[
\gamma_\alpha : A_\alpha \to A
\]

satisfying the following **universal property of direct sums**:

Given homomorphisms \( f_\alpha : A_\alpha \to K \) there is a unique \( f : A \to K \) s.t.

\[
\forall \alpha \ f_\alpha = f \circ \gamma_\alpha
\]

**Definition 91 (Direct sum in Chain)**

The direct sum for chain complexes \( \{ C^\alpha \} \) is the direct sum as abelian groups with the added stipulation that all homomorphisms must be chain maps
**Theorem 92 (Sing. simp. homology satisfies Axiom A4)**

If $X_\alpha$ are disjoint spaces. Then

$$H_n \left( \coprod X_\alpha \right) = \bigoplus_{\alpha} H_n(X_\alpha)$$

**Exercise 93 (HW4 - Problem 1)**

Verify that the homology functors $H_n : \text{Chain} \to \text{Ab}$ commute with direct sums.

In particular, show that if $C^\alpha$ are chain complexes then

$$H_n \left( \bigoplus_{\alpha} C^\alpha \right) \cong \bigoplus_{\alpha} H_n(C^\alpha)$$

This may be done by verifying that $H_n \left( \bigoplus_{\alpha} C^\alpha \right)$ satisfies the universal property for direct sums.
Definition 94 (Good pair)

\((X, A)\) is a **good pair** if there is a neighborhood \(V\) of \(A\) in \(X\) such that \(V\) deformation retracts to \(A\).

Theorem 95 (Relative homology and quotient spaces)

*If \((X, A)\) is a good pair then we have an isomorphism*

\[
q_* : H_n(X, A) \to \tilde{H}_n(X/A)
\]

where \(q_*\) is induced by the composition  
\[
C(X, A) \to C(X/A, A/A) \to \tilde{C}(X/A)
\]
Proof of Theorem 95.

\[
\begin{array}{ccc}
H_n(X, A) & \xrightarrow{\alpha} & H_n(X, V) \\
\downarrow{q_*} & & \downarrow \\
H_n(X/A, A/A) & \xrightarrow{\gamma} & H_n(X/A, V/A)
\end{array}
\]

\[
\begin{array}{ccc}
& & \xleftarrow{\beta} \\
& & \downarrow{q'_*} \\
& & \xleftarrow{\eta} \\
H_n(X - A, V - A) & \xrightarrow{\gamma^{-1}} & H_n(X/A - A/A, V/A - A/A)
\end{array}
\]

- $\beta$ and $\eta$ are isomorphisms by excision.
- The long exact sequence of the triple $(X, V, A)$ gives

\[
H_n(V, A) \to H_n(X, A) \xrightarrow{\alpha} H_n(X, V) \to H_{n-1}(V, A)
\]

Deformation retraction of $V$ to $A$ shows

\[
H_n(V, A) \cong H_n(A, A) \cong 0
\]

So $\alpha$ is an isomorphism.

- Same argument for $(X/A, V/A, A/A)$ shows $\gamma$ is isomorphism.
- $q'_*$ is an isomorphism since $(X - A, V - A) \to (X/A - A/A, V/A - A/A)$ is a homeo
- $q_* = \gamma^{-1}\eta q'_* \beta^{-1}\alpha$
**Corollary 96 (Long exact sequence for a good pair)**

If $(X, A)$ is a good pair then

$$\cdots \rightarrow \tilde{H}_n(A) \rightarrow \tilde{H}_n(X) \rightarrow \tilde{H}_n(X/A) \rightarrow \tilde{H}_{n-1}(A) \rightarrow \cdots$$

is exact.

**Proof.**

By Proposition 61 (Week 2) we have a long exact sequence

$$\cdots \rightarrow \tilde{H}_n(A) \rightarrow \tilde{H}_n(X) \rightarrow H_n(X, A) \rightarrow \tilde{H}_{n-1}(A) \rightarrow \cdots$$

By Theorem 95 above $H_n(X, A) \cong \tilde{H}(X/A)$
Proposition 97 (Homology groups of spheres)

\[ \tilde{H}_k(S^n) = \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n \end{cases} \]

Proof.

- True for \( S^{-1} = \emptyset \).
- True for \( S^0 = \{-1, 1\} \).
- Suppose true for \( S^{n-1} \).
- Consider good pair \((D^n, S^{n-1})\)

\[ \tilde{H}_k(D^n) \rightarrow \tilde{H}_k(D^n / S^{n-1}) \rightarrow \tilde{H}_{k-1}(S^{n-1}) \rightarrow \tilde{H}_{k-1}(D^n) \]

- \( D^n / S^{n-1} \cong S^n \)

\[ \tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1}) = \begin{cases} \mathbb{Z}, & k - 1 = n - 1 \\ 0, & k - 1 \neq n - 1 \end{cases} \]
Exercise 98 (HW4 - Problem 2)

*Show that for X nonempty*

\[ \tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX) \]

Exercise 99 (HW4 - Problem 3)

*Let X be the n-point set. Compute \( \tilde{H}_k(X) \)*

Exercise 100 (HW4 - Problem 4)

*Let \( Y = R^n = \bigvee_{i=1}^n S^1 \) be the rose with n-petals (see Hatcher pg. 10). Compute \( \tilde{H}_k(X) \)*
Definition 101 (Local homology groups)

\( x \in X \) the **local homology** of \( X \) at \( x \) is

\[
H_n(X, X - x)
\]

- Let \( V \) be any neighborhood of \( x \)
- By excision with \((X, A, U) = (X, X - x, X - V)\) we have

\[
H_n(X, X - x) = H_n(V, V - x)
\]

- Hence local homology depends only on topology near \( x \)
- Can use local homology show \( X \) not homeomorphic to \( Y \)
**Example 102**

$\mathbb{R}^n$ is contractible so the long exact sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n - 0)$ yeilds

\[ \tilde{H}_k(\mathbb{R}^n) \to H_k(\mathbb{R}^n, \mathbb{R}^n - 0) \to \tilde{H}_{k-1}(\mathbb{R}^n - 0) \to \tilde{H}_{k-1}(\mathbb{R}^n) \]

$\mathbb{R}^n - 0 \cong S^{n-1}$ so the local homology of $\mathbb{R}^n$ at 0 is

\[ H_k(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \tilde{H}_{k-1}(\mathbb{R}^n - 0) \cong \tilde{H}_{k-1}(S^{n-1}) = \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n \end{cases} \]

So $\mathbb{R}^n \ncong \mathbb{R}^m$ if $n \neq m$.

**Definition 103**

An *$n$-dimensional homology manifold* is a space $X$ such that for all $x \in X$

\[ H_k(X, X - x) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n \end{cases} \]
Naturality of the connecting homomorphism

- Suppose we have a commutative diagram of chain complexes and chain maps with exact rows.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & C & \rightarrow & D & \rightarrow & E & \rightarrow & 0 \\
\downarrow{\gamma} & & \circ & & \downarrow{\delta} & & \circ & & \downarrow{\varepsilon} \\
0 & \rightarrow & C' & \rightarrow & D' & \rightarrow & E' & \rightarrow & 0
\end{array}
\]

- The Snake Lemma gives

\[
\cdots \rightarrow H_{n+1}(E) \rightarrow H_n(C) \rightarrow H_n(D) \rightarrow H_n(E) \rightarrow H_{n-1}(C) \rightarrow \cdots
\]

\[
\begin{array}{cccccc}
\circ & & \varepsilon_* & & \circ & & \gamma_* & & \circ & & \delta_* & & \circ & & \varepsilon_* & & \gamma_* & & \circ \\
\downarrow & & \downarrow \varepsilon & & \downarrow {\gamma_*} & & \downarrow {\delta_*} & & \downarrow {\varepsilon_*} & & \downarrow \gamma_* & & \circ \\
\cdots \rightarrow H_{n+1}(E') \rightarrow H_n(C') \rightarrow H_n(D') \rightarrow H_n(E') \rightarrow H_{n-1}(C') \rightarrow \cdots
\end{array}
\]

- Fact that \( H_n : \text{Chain} \rightarrow \text{Ab} \) is a functor shows some squares commute.
Theorem 104 (Naturality of connecting homomorphism)

Given a commutative diagram of chain complexes and chain maps with exact rows.

\[
\begin{array}{cccccc}
0 & \rightarrow & C & \rightarrow & D & \rightarrow & E & \rightarrow & 0 \\
\gamma & \downarrow & \circ & \delta & \circ & \downarrow & \varepsilon & & \\
0 & \rightarrow & C' & \rightarrow & D' & \rightarrow & E' & \rightarrow & 0 \\
\end{array}
\]

The following diagram commutes:

\[
\begin{array}{ccc}
H_{n+1}(E) & \xrightarrow{\partial} & H_n(C) \\
\varepsilon_* & \circ & \gamma_* \\
H_{n+1}(E') & \xrightarrow{\partial} & H_n(C')
\end{array}
\]
Proof of Theorem 104.

We must verify that \( \partial \varepsilon_* = \gamma_* \partial \).

By definition \( \partial [e_n] = [c_{n-1}] \) where \( i_{n-1}(c_{n-1}) = \partial^D d_n \) for some \( d_n \in D_n \) with \( q_n(d_n) = e_n \).

Note that \( \varepsilon e_n = \varepsilon q_n(d_n) = q'_n \delta(d_n) \).

And \( \partial^D' \delta(d_n) = \delta \partial^D (d_n) = \delta i_{n-1}(c_{n-1}) = i'_{n-1} \gamma(c_{n-1}) \).

So \( \partial \varepsilon_* [e_n] = \partial [\varepsilon e_n] = [\gamma(c_{n-1})] = \gamma_* [c_{n-1}] = \gamma_* \partial [e_n] \).
Note on category theory:

**Definition 105 (Natural transformation)**

- Given two functors $F, G : \mathcal{C} \to \mathcal{D}$
- A **natural transformation** $\eta$ from $F$ to $G$ assigns a morphism $\eta_C : F(C) \to G(C)$ for each $C \in \text{Ob}(\mathcal{C})$ such that the following diagram commute for all morphisms $f \in \text{Mor}_\mathcal{C}(C_1, C_2)$

\[
\begin{array}{ccc}
F(C_1) & \xrightarrow{F(f)} & F(C_2) \\
\downarrow^{\eta_{C_1}} & \circ & \downarrow^{\eta_{C_2}} \\
G(C_1) & \xrightarrow{G(f)} & G(C_2)
\end{array}
\]
How does our naturality fit into this picture?

- Let $\mathcal{C} = \text{SESChain}$ be the category of short exact sequences of chain complexes $C \hookrightarrow D \twoheadrightarrow E$ with morphisms given by commuting diagrams of chain maps from one short exact sequence to another.

- Let $\mathcal{D} = \text{Ab}$
- Let $F(C \hookrightarrow D \twoheadrightarrow E) = H_n(E)$
- Let $G(C \hookrightarrow D \twoheadrightarrow E) = H_{n-1}(C)$
- Let $\eta(C \hookrightarrow D \twoheadrightarrow E) = \partial : H_n(E) \to H_{n-1}(C)$

Or

- Let $\mathcal{C} = \text{TopPair}$
- Let $\mathcal{D} = \text{Ab}$
- Let $F(X, A) = H_n(X, A)$
- Let $G(X, A) = H_{n-1}(A)$
- Let $\eta(X, A) = \partial : H_n(X, A) \to H_{n-1}(A)$
Exercise 106 (HW4 - Problem 5)

Show that if $P$ is the set of path components of $X$ then $H_0(X)$ is isomorphic to the free abelian group on $P$ and that if $X$ is nonempty then $\tilde{H}_0(X)$ is the free abelian group on $P - \{c_0\}$ where $c_0$ is a path component of $X$. 
Cellular Homology

Definition 107 (Chain complex for cellular homology)

Let $X$ be a CW complex (see Hatcher pg. 5). Let $X^{-1} = \emptyset$ and

$$C_n^{CW}(X) = \begin{cases} H_n(X^n, X^{n-1}), & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Let

$$\partial^{CW} : C_n^{CW}(X) \to C_{n-1}^{CW}(X)$$

be the connecting homomorphism

$$H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}, X^{n-2})$$

from the long exact sequence of the triple $(X^n, X^{n-1}, X^{n-2})$

Proposition 108

$C^{CW}(X)$ is a chain complex
Proof of Proposition 108.

- We must verify that \((\partial^{CW})^2 = 0\).
- We have the following commutative diagram of chain maps

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & C(X^n) & \longrightarrow & C(X^{n+1}) & \longrightarrow & C(X^{n+1}, X^n) & \longrightarrow & 0 \\
& & \downarrow j_\# & & \downarrow & & \downarrow 1 & & \\
0 & \longrightarrow & C(X^n, X^{n-1}) & \longrightarrow & C(X^{n+1}, X^{n-1}) & \longrightarrow & C(X^{n+1}, X^n) & \longrightarrow & 0
\end{array}
\]

- Naturality of the connecting homomorphism says the following square from the long exact sequence of homology commutes:

\[
\begin{array}{cccc}
H_{n+1}(X^{n+1}, X^n) & \longrightarrow & H_n(X^n) & \longrightarrow \\
\downarrow 1 & & \downarrow j_* & \\
H_{n+1}(X^{n+1}, X^n) & \longrightarrow & H_n(X^n, X^{n-1}) & \longrightarrow \\
\end{array}
\]

- So \(\partial^{CW} = j_* \partial\)
Proof of Proposition 108 (continued).

- We have commutative

$$
\begin{array}{ccc}
H_n(X^n) & \xrightarrow{j_*} & H_{n-1}(X^{n-1}, X^{n-2}) \\
\downarrow{\partial} & & \downarrow{\partial} \\
H_{n+1}(X^{n+1}, X^n) & \xrightarrow{\partial^{CW}} & H_n(X^n, X^{n-1}) & \xrightarrow{\partial^{CW}} & H_{n-1}(X^{n-1}, X^{n-2}) \\
\downarrow{\partial} & & \downarrow{\partial} \\
H_{n-1}(X^{n-1}) & & \\
\end{array}
$$

- Vertical composition

$$
H_n(X^n) \xrightarrow{j_*} H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1})
$$

is part of long exact sequence of the pair \((X^n, X^{n-1})\)

- So

$$(\partial^{CW})^2 = (j_* \partial) (j_* \partial) = j_* (\partial j_*) \partial = j_* \circ 0 \circ \partial = 0$$
Definition 109 (Cellular Homology)

If $X$ is a CW complex then the $n$th cellular homology group of $X$ is

$$H_n^{CW}(X) = H_n(C^{CW}(X))$$

For a CW complex $X$ how do $H_n^{CW}(X)$ and $H_n(X)$ relate?

Theorem 110 (Cellular and sing. simp. homology agree)

If $X$ is a CW complex then

$$H_n^{CW}(X) \cong H_n(X)$$

Advantages of $H^{CW}$

1. If $X$ is finite then $C^{CW}(X)$ is finitely generated
2. Boundary maps in $C^{CW}(X)$ can be easily understood.
The proof of Theorem 110 will use the following properties of homology for CW complexes.

**Lemma 111 (Homology of CW complexes)**

Let $X$ be a CW complex. Let $\Gamma^n$ be the set of $n$-cells of $X$

1. $H_k(X^n, X^{n-1}) \cong \begin{cases} \mathbb{Z}[\Gamma^n] & k = n \\ 0, & k \neq n \end{cases}$

2. If $k > n$
   
   $H_k(X^n) = 0$

3. If $k < n$ and $i : X^n \to X$ is the inclusion map then

   $H_k(X^n) \xrightarrow{i_*} H_k(X)$
Proof of Lemma 111 Claim 1.

- We have a commuting quotient maps of good pairs

\[
\bigg(\bigsqcup_{\alpha \in \Gamma_n} D^n, \bigsqcup_{\alpha \in \Gamma_n} S^{n-1}\bigg) \xrightarrow{r} (X^n, X^{n-1}) \xrightarrow{q} (X^n/X^{n-1}, \{\ast\})
\]

- By Theorem 95 we get that

\[
p_\ast : H_n\left(\bigsqcup_{\alpha \in \Gamma_n} D^n, \bigsqcup_{\alpha \in \Gamma_n} S^{n-1}\right) \rightarrow \tilde{H}_n(X^n/X^{n-1})
\]

and

\[
q_\ast : H_n(X^n, X^{n-1}) \rightarrow \tilde{H}_n(X^n/X^{n-1})
\]

are isomorphisms.

- So we get an isomorphism

\[
r_\ast : H_n\left(\bigsqcup_{\alpha \in \Gamma_n} D^n, \bigsqcup_{\alpha \in \Gamma_n} S^{n-1}\right) \rightarrow H_n(X^n, X^{n-1})
\]
Proof of Lemma 111 Claim 1 (continued).

- Applying the additivity axiom

\[ H_n(X^n, X^{n-1}) \cong H_n \left( \coprod_{\alpha \in \Gamma^n} D^n, \coprod_{\alpha \in \Gamma^n} S^{n-1} \right) \]

\[ \cong \bigoplus_{\alpha \in \Gamma^n} H_n(D^n, S^{n-1}) \]

\[ \cong \bigoplus_{\alpha \in \Gamma^n} \tilde{H}_n(S^n) \]

\[ \cong \bigoplus_{\alpha \in \Gamma^n} \mathbb{Z} \]

\[ \cong \mathbb{Z}[\Gamma^n] \]