Math 757
Homology theory

January 27, 2011

Axiom 4: Additivity

Definition 90 (Direct sum in **Ab**)

Given abelian groups $\{A_\alpha\}$, the **direct sum** is an abelian group A with homomorphisms

$$\gamma_\alpha : A_\alpha \to A$$

satifying the following **universal property of direct sums:**

Given homomorphisms $f_\alpha : A_\alpha \to K$ there is a unique $f : A \to K$ s.t. $\forall \alpha \, f_\alpha = f \circ \gamma_\alpha$

Definition 91 (Direct sum in **Chain**)

The direct sum for chain complexes $\{C^\alpha\}$ is the direct sum as abelian groups with the added stipulation that all homomorphisms must be chain maps

Theorem 92 (Sing. simp. homology satisfies Axiom A4)

X_α disjoint spaces. Then

$$H_n\left(\coprod X_\alpha\right) = \bigoplus_\alpha H_n(X_\alpha)$$

Exercise 93 (HW4 - Problem 1)

*Verify that the homology functors H_n : **Chain** $\to$ **Ab** commute with direct sums.*

In particular, show that if C^α are chain complexes then

$$H_n\left(\bigoplus_\alpha C^\alpha\right) \cong \bigoplus_\alpha H_n\left(C^\alpha\right)$$

This may be done by verifying that $H_n\left(\bigoplus_\alpha C^\alpha\right)$ satisfies the universal property for direct sums.

Definition 94 (Good pair)

(X, A) is a **good pair** if there is a neighborhood V of A in X such that V deformation retracts to A.

Theorem 95 (Relative homology and quotient spaces)

If (X, A) is a good pair then we have an isomorphism

$$q_* : H_n(X, A) \to \tilde{H}_n(X/A)$$

where q_* is induced by the composition
$C(X, A) \to C(X/A, A/A) \to \tilde{C}(X/A)$

Proof of Theorem 95.

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\alpha"] \arrow[d, "q_*"'] & H_n(X,V) \arrow[d] & H_n(X-A,V-A) \arrow[l, "\beta"'] \arrow[d, "q'_*"'] \\
H_n(X/A,A/A) \arrow[r, "\gamma"] & H_n(X/A,V/A) & H_n(X/A-A/A,V/A-A/A) \arrow[l, "\eta"']
\end{tikzcd}$$

- β and η are isomorphisms by excision.
- The long exact sequence of the triple (X,V,A) gives

$$H_n(V,A) \to H_n(X,A) \xrightarrow{\alpha} H_n(X,V) \to H_{n-1}(V,A)$$

  Deformation retaction of V to A shows

$$H_n(V,A) \cong H_n(A,A) \cong 0$$

  So α is an isomorphism
- Same argument for $(X/A, V/A, A/A)$ shows γ is isomorphism.
- q'_* is an isomorphism since $(X-A, V-A) \to (X/A - A/A, V/A - A/A)$ is a homeo
- $q_* = \gamma^{-1}\eta q'_* \beta^{-1}\alpha$ $\square$

Corollary 96 (Long exact sequence for a good pair)

If (X, A) is a good pair then

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to \cdots$$

is exact.

Proof.

By Proposition 61 (Week 2) we have a long exact sequence

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X, A) \to \tilde{H}_{n-1}(A) \to \cdots$$

By Theorem 95 above $H_n(X, A) \cong \tilde{H}(X/A)$ $\square$

Proposition 97 (Homology groups of spheres)

$$\tilde{H}_k(S^n) = \begin{cases} \mathbf{Z}, & k = n \\ 0, & k \neq n \end{cases}$$

Proof.

- True for $S^{-1} = \varnothing$.
- True for $S^0 = \{-1, 1\}$.
- Suppose true for S^{n-1}
- Consider good pair (D^n, S^{n-1})

$$\tilde{H}_k(D^n) \to \tilde{H}_k(D^n/S^{n-1}) \to \tilde{H}_{k-1}(S^{n-1}) \to \tilde{H}_{k-1}(D^n)$$

- $D^n/S^{n-1} \cong S^n$
- $$\tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1}) = \begin{cases} \mathbf{Z}, & k-1 = n-1 \\ 0, & k-1 \neq n-1 \end{cases}$$

$\square$

Exercise 98 (HW4 - Problem 2)

Show that for X nonempty

$$\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$$

Exercise 99 (HW4 - Problem 3)

Let X be the n-point set. Compute $\tilde{H}_k(X)$

Exercise 100 (HW4 - Problem 4)

Let $Y = R^n = \bigvee_{i=1}^n S^1$ be the rose with n-petals (see Hatcher pg. 10). Compute $\tilde{H}_k(X)$

Definition 101 (Local homology groups)

$x \in X$ the **local homology** of X at x is

$$H_n(X, X - x)$$

- Let V be any neighborhood of x
- By excision with $(X, A, U) = (X, X - x, X - V)$ we have

$$H_n(X, X - x) = H_n(V, V - x)$$

- Hence local homology depends only on topology near x
- Can use local homology show X not homeomorphic to Y

Example 102

$\mathbf{R}^n$ is contractible so the long exact sequence of the pair $(\mathbf{R}^n, \mathbf{R}^n - 0)$ yeilds

$$\tilde{H}_k(\mathbf{R}^n) \to H_k(\mathbf{R}^n, \mathbf{R}^n - 0) \to \tilde{H}_{k-1}(\mathbf{R}^n - 0) \to \tilde{H}_{k-1}(\mathbf{R}^n)$$

$\mathbf{R}^n - 0 \simeq S^{n-1}$ so the local homology of $\mathbf{R}^n$ at 0 is

$$H_k(\mathbf{R}^n, \mathbf{R}^n - 0) \cong \tilde{H}_{k-1}(\mathbf{R}^n - 0) \cong \tilde{H}_{k-1}(S^{n-1}) = \begin{cases} \mathbf{Z}, & k = n \\ 0, & k \neq n \end{cases}$$

So $\mathbf{R}^n \ncong \mathbf{R}^m$ if $n \neq m$.

Definition 103

An n-dimensional **homology manifold** is a space X such that for all $x \in X$

$$H_k(X, X - x) \cong \begin{cases} \mathbf{Z}, & k = n \\ 0, & k \neq n \end{cases}$$

Naturality of the connecting homomorphism

- Suppose we have a commutative diagram of chain complexes and chain maps with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d, "\gamma"] \arrow[dr, phantom, "\circlearrowleft"] & D \arrow[r] \arrow[d, "\delta"] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \arrow[r] & 0
\end{tikzcd}$$

- The Snake Lemma gives

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(E) \arrow[r] \arrow[d, "\varepsilon_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(C) \arrow[r] \arrow[d, "\gamma_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(D) \arrow[r] \arrow[d, "\delta_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(E) \arrow[r] \arrow[d, "\varepsilon_*"] & H_{n-1}(C) \arrow[r] \arrow[d, "\gamma_*"] \arrow[dr, phantom, "\circlearrowleft"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(E') \arrow[r] & H_n(C') \arrow[r] & H_n(D') \arrow[r] & H_n(E') \arrow[r] & H_{n-1}(C') \arrow[r] & \cdots
\end{tikzcd}$$

- Fact that $H_n : \mathbf{Chain} \to \mathbf{Ab}$ is a functor shows some squares commute.

Theorem 104 (Naturality of connecting homomorphism)

Given a commutative diagram of chain complexes and chain maps with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d, "\gamma"] \arrow[dr, phantom, "\circlearrowleft"] & D \arrow[r] \arrow[d, "\delta"] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \arrow[r] & 0
\end{tikzcd}$$

The following diagram commutes:

$$\begin{tikzcd}
H_{n+1}(E) \arrow[r, "\partial"] \arrow[d, "\varepsilon_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(C) \arrow[d, "\gamma_*"] \\
H_{n+1}(E') \arrow[r, "\partial"] & H_n(C')
\end{tikzcd}$$

Proof of Theorem 104.

$$\begin{tikzcd}
0 \arrow[r] & C_n \arrow[r, "i_n"] \arrow[d, "\partial_n^C"] & D_n \arrow[r, "q_n"] \arrow[d, "\partial_n^D"] & E_n \arrow[r] \arrow[d, "\partial_n^E"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r, "i_{n-1}"] & D_{n-1} \arrow[r, "q_{n-1}"] & E_{n-1} \arrow[r] & 0
\end{tikzcd}$$

- We must verify that $\partial\varepsilon_* = \gamma_*\partial$.
- By definition $\partial[e_n] = [c_{n-1}]$ where $i_{n-1}(c_{n-1}) = \partial_n^D d_n$ for some $d_n \in D_n$ with $q_n(d_n) = e_n$
- Note that $\varepsilon e_n = \varepsilon q_n(d_n) = q_n'\delta(d_n)$
- And $\partial_n^{D'}\delta(d_n) = \delta\partial_n^D(d_n) = \delta i_{n-1}(c_{n-1}) = i_{n-1}'\gamma(c_{n-1})$
- so $\partial\varepsilon_*[e_n] = \partial[\varepsilon e_n] = [\gamma(c_{n-1})] = \gamma_*[c_{n-1}] = \gamma_*\partial[e_n]$

$\square$

Note on category theory:

Definition 105 (Natural transformation)

- Given two functors $F, G : \mathcal{C} \to \mathcal{D}$
- A **natural transformation** η from F to G assigns a morphism $\eta_C : F(C) \to G(C)$ for each $C \in \mathrm{Ob}(\mathcal{C})$ such that the following diagram commute for all morphisms $f \in \mathrm{Mor}_{\mathcal{C}}(C_1, C_2)$

$$\begin{tikzcd} F(C_1) \arrow[r, "F(f)"] \arrow[d, "\eta_{C_1}"] \arrow[dr, phantom, "\circlearrowleft"] & F(C_2) \arrow[d, "\eta_{C_2}"] \\ G(C_1) \arrow[r, "G(f)"] & G(C_2) \end{tikzcd}$$

How does our naturality fit into this picture?

- Let $\mathcal{C} = \mathbf{SESChain}$ be the category of short exact sequences of chain complexes $C \hookrightarrow D \twoheadrightarrow E$ with morphisms given by commuting diagrams of chain maps from one short exact sequence to another.
- Let $\mathcal{D} = \mathbf{Ab}$
- Let $F(C \hookrightarrow D \twoheadrightarrow E) = H_n(E)$
- Let $G(C \hookrightarrow D \twoheadrightarrow E) = H_{n-1}(C)$
- Let $\eta_{(C \hookrightarrow D \twoheadrightarrow E)} = \partial : H_n(E) \to H_{n-1}(C)$

Or

- Let $\mathcal{C} = \mathbf{TopPair}$
- Let $\mathcal{D} = \mathbf{Ab}$
- Let $F(X, A) = H_n(X, A)$
- Let $G(X, A) = H_{n-1}(A)$
- Let $\eta_{(X,A)} = \partial : H_n(X, A) \to H_{n-1}(A)$

Exercise 106 (HW4 - Problem 5)

Show that if P is the set of path components of X then $H_0(X)$ is isomorphic to the free abelian group on P and that if X is nonempty then $\tilde{H}_0(X)$ is the free abelian group on $P - \{c_0\}$ where c_0 is a path component of X.

Cellular Homology

Definition 107 (Chain complex for cellular homology)

Let X be a CW complex (see Hatcher pg. 5). Let $X^{-1} = \varnothing$ and

$$C_n^{\mathrm{CW}}(X) = \begin{cases} H_n(X^n, X^{n-1}), & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Let

$$\partial^{\mathrm{CW}} : C_n^{\mathrm{CW}}(X) \to C_{n-1}^{\mathrm{CW}}(X)$$

be the connecting homomorphism

$$H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}, X^{n-2})$$

from the long exact sequence of the triple (X^n, X^{n-1}, X^{n-2})

Proposition 108

$C^{\mathrm{CW}}(X)$ is a chain complex

Proof of Proposition 108.

- We must verify that $(\partial^{\mathrm{CW}})^2 = 0$.
- We have the following commutative diagram of chain maps

$$\begin{tikzcd}
0 \arrow[r] & C(X^n) \arrow[r] \arrow[d, "j_\sharp"] & C(X^{n+1}) \arrow[r] \arrow[d] & C(X^{n+1}, X^n) \arrow[r] \arrow[d, "\mathbf{1}"] & 0 \\
0 \arrow[r] & C(X^n, X^{n-1}) \arrow[r] & C(X^{n+1}, X^{n-1}) \arrow[r] & C(X^{n+1}, X^n) \arrow[r] & 0
\end{tikzcd}$$

- Naturality of the connecting homomorphism says the following square from the long exact sequence of homology commutes:

$$\begin{tikzcd}
H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial"] \arrow[d, "\mathbf{1}"] & H_n(X^n) \arrow[d, "j_*"] \\
H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial^{\mathrm{CW}}"] & H_n(X^n, X^{n-1})
\end{tikzcd}$$

- So $\partial^{\mathrm{CW}} = j_* \partial$

Proof of Proposition 108 *(continued)*.

- We have commutative

$$\begin{tikzcd}
 & H_n(X^n) \arrow[d, "j_*"] & \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial"] \arrow[r, "\partial^{\mathrm{CW}}"] & H_n(X^n, X^{n-1}) \arrow[r, "\partial^{\mathrm{CW}}"] \arrow[d, "\partial"] & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & H_{n-1}(X^{n-1}) \arrow[ur, "j_*"] &
\end{tikzcd}$$

- Vertical composition

$$H_n(X^n) \xrightarrow{j_*} H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1})$$

  is part of long exact sequence of the pair (X^n, X^{n-1})

- So

$$(\partial^{\mathrm{CW}})^2 = (j_*\partial)(j_*\partial) = j_*(\partial j_*)\partial = j_* \circ 0 \circ \partial = 0$$

$\square$

Definition 109 (Cellular Homology)

If X is a CW complex then the n**th cellular homology group** of X is

$$H_n^{\mathrm{CW}}(X) = H_n(C^{\mathrm{CW}}(X))$$

For a CW complex X how do $H_n^{\mathrm{CW}}(X)$ and $H_n(X)$ relate?

Theorem 110 (Cellular and sing. simp. homology agree)

If X is a CW complex then

$$H_n^{\mathrm{CW}}(X) \cong H_n(X)$$

Advantages of H^{CW}

1. If X is finite then $C^{\mathrm{CW}}(X)$ is finitely generated
2. boundary maps in $C^{\mathrm{CW}}(X)$ can be easily understood.

The proof of Theorem 110 will use the following properties of homology for CW complexes.

Lemma 111 (Homology of CW complexes)

Let X is a CW complex. Let Γ^n be the set of n-cells of X

1. $$H_k(X^n, X^{n-1}) \cong \begin{cases} \mathbf{Z}[\Gamma^n] & k = n \\ 0, & k \neq n \end{cases}$$
2. *If $k > n$*
$$H_k(X^n) = 0$$
3. *If $k < n$ and $i : X^n \to X$ is the inclusion map then*
$$H_k(X^n) \xrightarrow[i_*]{\cong} H_k(X)$$

Proof of Lemma 111 Claim 1.

- We have a commuting quotient maps of good pairs

$$\begin{tikzcd} (\coprod_{\alpha\in\Gamma^n} D^n, \coprod_{\alpha\in\Gamma^n} S^{n-1}) \arrow[r, "r"] \arrow[rr, bend left, "p"] & (X^n, X^{n-1}) \arrow[r, "q"] & (X^n/X^{n-1}, \{*\}) \end{tikzcd}$$

- By Theorem 95 we get that

$$p_* : H_n\left(\coprod_{\alpha\in\Gamma^n} D^n, \coprod_{\alpha\in\Gamma^n} S^{n-1}\right) \to \tilde{H}_n(X^n/X^{n-1})$$

  and

$$q_* : H_n(X^n, X^{n-1}) \to \tilde{H}_n(X^n/X^{n-1})$$

  are isomorphisms.

- So we get an isomorphism

$$r_* : H_n\left(\coprod_{\alpha\in\Gamma^n} D^n, \coprod_{\alpha\in\Gamma^n} S^{n-1}\right) \to H_n(X^n, X^{n-1})$$

Proof of Lemma 111 Claim 1 *(continued)*.

- Applying the additivity axiom

$$\begin{aligned} H_n(X^n, X^{n-1}) &\cong H_n\left(\coprod_{\alpha\in\Gamma^n} D^n, \coprod_{\alpha\in\Gamma^n} S^{n-1}\right) \\ &\cong \bigoplus_{\alpha\in\Gamma^n} H_n(D^n, S^{n-1}) \\ &\cong \bigoplus_{\alpha\in\Gamma^n} \tilde{H}_n(S^n) \\ &\cong \bigoplus_{\alpha\in\Gamma^n} \mathbf{Z} \\ &\cong \mathbf{Z}[\Gamma^n] \end{aligned}$$

$\square$