Math 757
Homology theory

February 10, 2011

Definition 109 (Cellular Homology)

If X is a CW complex then the *n***th cellular homology group** of X is

$$H_n^{\mathrm{CW}}(X) = H_n(C^{\mathrm{CW}}(X))$$

For a CW complex X how do $H_n^{\mathrm{CW}}(X)$ and $H_n(X)$ relate?

Theorem 110 (Cellular and sing. simp. homology agree)

If X is a CW complex then

$$H_n^{\mathrm{CW}}(X) \cong H_n(X)$$

Advantages of H^{CW}

1. If X is finite then $C^{\mathrm{CW}}(X)$ is finitely generated
2. boundary maps in $C^{\mathrm{CW}}(X)$ can be easily understood.

The proof of Theorem 110 will use the following properties of homology for CW complexes.

Lemma 111 (Homology of CW complexes)

Let X be a CW complex. Let Γ^n be the set of n-cells of X

1. $$H_k(X^n, X^{n-1}) \cong \begin{cases} \mathbf{Z}[\Gamma^n] & k = n \\ 0, & k \neq n \end{cases}$$
2. *If $k > n$*
$$H_k(X^n) = 0$$
3. *If $k < n$ and $i : X^n \to X$ is the inclusion map then*
$$H_k(X^n) \xrightarrow[i_*]{\cong} H_k(X)$$

Proof of Lemma 111 Claim 1.

- We have a commuting quotient maps of good pairs

$$\begin{tikzcd}
\left(\coprod_{\alpha\in\Gamma^n} D^n, \coprod_{\alpha\in\Gamma^n} S^{n-1}\right) \arrow[r, "r"] \arrow[rr, bend left, "p"] & (X^n, X^{n-1}) \arrow[r, "q"] & (X^n/X^{n-1}, \{*\})
\end{tikzcd}$$

- By Theorem 95 we get that

$$p_* : H_k\left(\coprod_{\alpha\in\Gamma^n} D^n, \coprod_{\alpha\in\Gamma^n} S^{n-1}\right) \to \tilde{H}_k(X^n/X^{n-1})$$

  and

$$q_* : H_k(X^n, X^{n-1}) \to \tilde{H}_k(X^n/X^{n-1})$$

  are isomorphisms.

- So we get an isomorphism

$$r_* : H_k\left(\coprod_{\alpha\in\Gamma^n} D^n, \coprod_{\alpha\in\Gamma^n} S^{n-1}\right) \to H_k(X^n, X^{n-1})$$

Proof of Lemma 111 Claim 1 *(continued)*.

- Hence

$$\begin{aligned}
H_k(X^n, X^{n-1}) &\cong H_k\left(X^n/X^{n+1}\right) \quad \text{(Thm 95)} \\
&\cong H_k\left(\coprod_{\alpha\in\Gamma^n} D^n, \coprod_{\alpha\in\Gamma^n} S^{n-1}\right) \quad \text{(Thm 95)} \\
&\cong \bigoplus_{\alpha\in\Gamma^n} H_k(D^n, S^{n-1}) \quad \text{(A4. Additivity)} \\
&\cong \bigoplus_{\alpha\in\Gamma^n} \tilde{H}_k(S^n) \\
&\cong \begin{cases} \bigoplus_{\alpha\in\Gamma^n} \mathbf{Z} & k = n \\ 0, & k \neq n \end{cases} \\
&\cong \begin{cases} \mathbf{Z}[\Gamma^n] & k = n \\ 0, & k \neq n \end{cases}
\end{aligned}$$

□

Proof of Lemma 111 Claim 2.

Now we show Claim 2: If $k > n$

$$H_k(X^n) = 0$$

Assume $k > n$

- In the long exact sequence for the pair (X^n, X^{n-1}) we have

$$\overbrace{H_{k+1}(X^n, X^{n-1})}^{0} \to H_k(X^{n-1}) \to H_k(X^n) \to \overbrace{H_k(X^n, X^{n-1})}^{0}$$

- So

$$H_k(X^n) \cong H_k(X^{n-1})$$

- $k > n - 1 > n - 2 > \cdots > 1$ so

$$H_k(X^n) \cong H_k(X^{n-1}) \cong \cdots \cong H_k(X^0) \cong 0$$

□

Proof of Lemma 111 Claim 3.

Now we show Claim 3: If $k < n$

$$H_k(X^n) = H_n(X)$$

Assume $k < n$

- Similarly, long exact sequence for the pair (X^{n+1}, X^n) gives

$$\overbrace{H_{k+1}(X^{n+1}, X^n)}^{0} \to H_k(X^n) \to H_k(X^{n+1}) \to \overbrace{H_k(X^{n+1}, X^n)}^{0}$$

- So

$$H_k(X^n) \cong H_k(X^{n+1})$$

- $k < n < n+1 < n+2 < \cdots$ so

$$H_k(X^n) \cong H_k(X^{n+1}) \cong H_k(X^{n+2}) \cong \cdots$$

- **IF** X finite dimensional then $X = X^{n+m}$ for some m and we get

$$H_k(X^n) \cong H_k(X^{n+m}) \cong H_k(X)$$

Proof of Lemma 111 Claim 3 *(continued)*.

- Claim: $H_k(X^n) \xrightarrow{i_*} H_k(X)$ is surjective
  - Let $[c] \in H_k(X)$ be a cycle. Where $c = \sum_{i=1}^{\ell} \sigma_i^k$
  - $\bigcup_{i=1}^{\ell} \sigma_i^k(\Delta^k)$ is compact so by Hatcher Prop A.1 there is some m such that
$$\bigcup_{i=1}^{\ell} \sigma_i^k(\Delta^k) \subset X^{n+m}$$
  - Thus $[c]$ is in the image of $H_k(X^{n+m}) \xrightarrow{i'_*} H_k(X)$
  - Thus $[c]$ is in the image of the composition
$$H_k(X^n) \cong H_k(X^{n+m}) \xrightarrow{i'_*} H_k(X)$$

- Claim: $H_k(X^n) \xrightarrow{i_*} H_k(X)$ is injective
  - Suppose $[c] = 0$ in $H_k(X)$
  - Then $c = \partial d$ for some chain $d \in C_{k+1}(X)$
  - There is m s.t. $d \in C_{k+1}(X^{n+m})$ so $[c] = 0$ in $H_k(X^{n+m}) \cong H_k(X^n)$

□

Proof of Theorem 110.

$$\begin{tikzcd}
 & & & & 0 \\
 & & 0 \arrow[d] & H_n(X^{n+1}) \cong H_n(X) \arrow[ur] & \\
 & & H_n(X^n) \arrow[ur] \arrow[d, "j_n"] & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial^{\mathrm{CW}}_{n+1}"'] \arrow[ur, "\partial_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[r, "\partial^{\mathrm{CW}}_n"] \arrow[d, "\partial_n"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & & \\
 & 0 \arrow[ur] & & &
\end{tikzcd}$$

- $H_n(X) \cong \frac{H_n(X^n)}{\partial_{n+1} H_{n+1}(X^{n+1}, X^n)}$
- $\partial^{\mathrm{CW}}_{n+1} H_{n+1}(X^{n+1}, X^n) = j_n \partial_{n+1} H_{n+1}(X^{n+1}, X^n) \cong \partial_{n+1} H_{n+1}(X^{n+1}, X^n)$
- $\ker \partial^{\mathrm{CW}}_n = \ker \partial_n = j_{n-1} H_n(X^n) \cong H_n(X^n)$
- $$H_n(X) \cong \frac{H_n(X^n)}{\partial_{n+1} H_{n+1}(X^{n+1}, X^n)} \xrightarrow[j_n]{\cong} \frac{\ker \partial^{\mathrm{CW}}_n}{\partial^{\mathrm{CW}}_{n+1} H_n(X^n, X^{n-1})} = H^{\mathrm{CW}}_n(X)$$

$\square$

Classical invariants

Definition 112 (Torsion and rank)

For an abelian group A the **torsion subgroup** is

$$A_{\text{tor}} = \left\{ a \in A \middle| \exists n \in \mathbf{Z} \text{ s.t. } na = 0 \right\}$$

There is a unique cardinal n such that

$$A/A_{\text{tor}} \cong \oplus^n \mathbf{Z}$$

The **rank** of A is

$$\operatorname{rank} A = n$$

Lemma 113 (Rank-nullity Theorem)

$B \subset A$ abelian groups then

$$\operatorname{rank} B + \operatorname{rank} A/B = \operatorname{rank} A$$

Definition 114 (Betti number)

For a space X the k**th Betti number** is

$$b_k(X) = \operatorname{rank} H_k(X)$$

Definition 115 (Euler characteristic)

If $H(X)$ has finite rank the **Euler characteristic** is

$$\chi(X) = \sum_k (-1)^k b_k(X)$$

- We see immediately that Euler characteristic and Betti numbers are invariants of homotopy type
- $b_0(X)$ is the number of path components of X

Example 116 (Invariants of spheres)

The Betti numbers for S^n are

$$b_k(S^n) = \begin{cases} 1, & k \in \{0, n\} \\ 0, & k \notin \{0, n\} \end{cases}$$

The Euler characteristic for S^n is

$$\chi(S^n) = 1 + (-1)^n = \begin{cases} 2, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Definition 117 (Finite CW complex)

A CW complex is **finite** if it has finitely many cells.

Proposition 118 (Euler characteristic of a CW complex)

Given a finite CW complex $X = X^n$ let α_k be the number of k-cells of X.

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \alpha_k$$

Exercise 119 (HW5 - Problem 1)

Prove Proposition 118

Exercise 120 (HW5 - Problem 2)

Show that if X and Y are finite CW complexes then

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y)$$

Degree and local degree

Definition 121 (Degree)

Let $f : S^n \to S^n$ be a map where $n > 0$. Then

$$f_* : H_n(S^n) \to H_n(S^n)$$

is a homomorphism of $\mathbf{Z}$ to $\mathbf{Z}$ which must be of the form

$$f_*(\alpha) = d \cdot \alpha$$

for some $d \in \mathbf{Z}$. The **degree** of f is

$$\deg f = d$$

- The identity map $\mathbf{1} : S^n \to S^n$ has degree 1 by functoriality of H_*
- $\deg(f \circ g) = (\deg f) \cdot (\deg g)$ by functoriality of H_*
- If f has an inverse (or just a homotopy inverse) then $\deg f = \pm 1$

Definition 122 (Local degree)

Suppose

- $f : S^n \to S^n$ is a map sending $x \in S^n$ to $y \in S^n$
- x has a neighborhood U such that $y \notin f(U - x)$
- Let $V = f(U)$.

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "\cong"'] \arrow[rrrrr, bend left, "\varphi_x"] & H_n(S^n, S^n - x) \arrow[r, "\text{exc.}", "\cong"'] & H_n(U, U - x) \arrow[r, "f_*"] & H_n(V, V - y) & H_n(S^n, S^n - y) \arrow[l, "\text{exc.}"', "\cong"] & H_n(S^n) \arrow[l, "\cong"]
\end{tikzcd}$$

There must be some $d_x \in \mathbf{Z}$ such that

$$\varphi_x(\alpha) = d_x \cdot \alpha$$

The **local degree** of f at x is

$$\deg_x f = d_x$$

- If f is a local homeomorphism near x then $\deg_x f = \pm 1$

- The following proposition gives an effective method of determining the degree of many maps $f : S^n \to S^n$

Proposition 123 (Degree from local degree)

Given $f : S^n \to S^n$ if there is $y \in S^n$ such that

$$f^{-1}(y) = \{x_1, \cdots, x_m\}$$

is a finite set then

$$\deg f = \sum_{i=1}^{m} \deg_{x_i} f$$

Proof of Proposition 123.

Suppose

- $$f^{-1}(y) = \{x_1, \cdots, x_m\}$$
- y has a neighborhood V
- x_i's have disjoint neighborhoods U_i such that $f(U_i) \subset V$

Then we have commutative

$$\begin{tikzcd}
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "b_{i*}"', "\cong"] \arrow[d, "k_i"] & H_n(V, V - y) \arrow[d, "c_*", "\cong"'] \\
H_n(S^n, S_n - x_i) & \begin{array}{c} \oplus_i H_n(U_i, U_i - x_i) \cong \\ H_n(S^n, S^n - \{x_1, \cdots, x_m\}) \end{array} \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[ul, "a_{i*}", "\cong"'] \arrow[u, "j_*"'] & H_n(S^n) \arrow[u, "e_*"', "\cong"]
\end{tikzcd}$$

Proof of Proposition 123 *(continued)*.

$$\begin{tikzcd}
& H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "b_i"', "\cong"] \arrow[d, "k_i"] & H_n(V, V - y) \arrow[d, "\cong"', "c"] \\
H_n(S^n, S_n - x_i) & \begin{array}{c} \oplus_i H_n(U_i, U_i - x_i) \cong \\ H_n(S^n, S^n - \{x_1, \cdots, x_m\}) \end{array} \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
& H_n(S^n) \arrow[r, "f_*"] \arrow[ul, "\cong", "a_i"'] \arrow[u, "j"'] & H_n(S^n) \arrow[u, "\cong", "e"']
\end{tikzcd}$$

For $1 \in H_n(S^n) \cong \mathbf{Z}$

- $(\deg_{x_i} f) \cdot 1 = (e^{-1} c f_* b_i^{-1} a_i)(1)$
- $(\deg f) \cdot 1 = (e^{-1} f_* j)(1)$
- $k_i(1) = (0, \cdots, 0, \overbrace{1}^{i}, 0, \cdots, 0)$
- $1 = b_i(1) = p_i k_i(1) = p_i(0, \cdots, 0, \overbrace{1}^{i}, 0, \cdots, 0)$

Proof of Proposition 123 *(continued)*.

$$\begin{tikzcd}
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "b_i"', "\cong"] \arrow[d, "k_i"] & H_n(V, V - y) \arrow[d, "c", "\cong"'] \\
H_n(S^n, S_n - x_i) & \bigoplus_i H_n(U_i, U_i - x_i) \cong H_n(S^n, S^n - \{x_1, \cdots, x_m\}) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[ul, "a_i", "\cong"'] \arrow[u, "j"'] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "e"', "\cong"]
\end{tikzcd}$$

- For all i we have $1 = a_i(1) = p_i j(1)$
- $j(1) = (1, \cdots, 1) = \sum_{i=1}^m (0, \cdots, 0, \overbrace{1}^{i}, 0, \cdots, 0) = \sum_{i=1}^m k_i(1)$
- $\deg_{x_i} f = e^{-1} c f_* b_i^{-1} a_i(1) = e^{-1} f_* k_i(1) =$
  $e^{-1} f_* (0, \cdots, 0, \overbrace{1}^{i}, 0, \cdots, 0)$
- $\deg f = e^{-1} f_* j(1) = e^{-1} f_* (1, \cdots, 1) = \sum_{i=1}^m \deg_{x_i} f$

☐

Example 124 (Selfmaps of S^1)

- Let $S^1 = \left\{ z \in \mathbf{C} \middle| |z| = 1 \right\}$
- Let $f_n : S^1 \to S^1$ be the map $f(z) = z^n$
- Claim: $\deg f_n = n$
- $f^{-1}(1) = \left\{ z_k = e^{\frac{2k\pi i}{n}} \in \mathbf{C} \middle| |z| = 1 \right\}$
- May homotope f so that in a neighborhood of each z_k f is a rotation.
- $\deg_{z_k} f = 1$
- $\deg f = \sum_{k=1}^{n} \deg_{z_k} f = \sum_{k=1}^{n} 1 = n$

Proposition 125

Let $Sf : S^{n+1} \to S^{n+1}$ be the suspension of $f : S^n \to S^n$. Then

$$\deg Sf = \deg f$$

Degree and the cellular boundary map

X a CW complex. Recall

- $$\Gamma^n = \{\ n\text{-cells of } X\ \}$$
- $$C_n^{\mathrm{CW}}(X) = H_n(X^n, X^{n-1}) \cong \mathbf{Z}[\Gamma^n]$$
- $$\partial^{\mathrm{CW}} = H_k(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

  Is the connecting homomorphism of the triple (X^n, X^{n-1}, X^{n-2})

Proposition 126 (Cellular boundary formula)

- *X CW complex*
- *Let $\{e^n_\alpha\}$ be the n-cells of X*
- *Let $\left\{e^{n-1}_\beta\right\}$ be the $(n-1)$-cells of X*
- *Let $d_{\alpha\beta} = \deg \varphi_{\alpha\beta}$ where*

$$\varphi_{\alpha\beta} : S^{n-1}_\alpha \to S^{n-1}_\beta$$

is the composition

$$S^{n-1}_\alpha = \partial D_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \to \frac{X^{n-1}}{X^{n-1} - e^{n-1}_\beta} = S^{n-1}_\beta$$

Then

$$\partial^{\mathrm{CW}}(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$$

Proof of Proposition 126.

$$\begin{tikzcd}
H_n(D_\alpha, \partial D^n_\alpha) \arrow[r, "\partial", "\cong"'] \arrow[d, "\Phi_{\alpha *}"'] & \tilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & \tilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "\partial_n^{\mathrm{CW}}"'] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta *}"'] \arrow[d, "\cong"] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\cong"] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{tikzcd}$$

- $\Phi_\alpha : D_\alpha \to X^n$ the characteristic map
- $\varphi_\alpha : \partial D_\alpha \to X^{n-1}$ the attaching map
- $q : X^{n-1} \to X^{n-1}/X^{n-2}$ the quotient map
- $q_\beta : X^{n-1}/X^{n-2} \to S^{n-1}_\beta$ the quotient map
- $\Delta_{\alpha\beta} = q_\beta q \varphi_\alpha$
- upper left and lower left commutative by naturality. upper and lower right by comm. chain maps.

Proof of Proposition 126 *(continued)*.

$$\begin{tikzcd}
H_n(D_\alpha, \partial D^n_\alpha) \arrow[r, "\partial", "\cong"'] \arrow[d, "\Phi_{\alpha *}"'] & \tilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & \tilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "\partial^{\mathrm{CW}}_n"'] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta *}"'] \arrow[d, "\cong"] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\cong"] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{tikzcd}$$

- $\tilde{H}_{n-1}(X^{n-1}/X^{n-2}) = \bigoplus_\beta q^{-1}_{\beta *}(\tilde{H}_{n-1}(S^{n-1}_\beta))$ (Lemma 110 Claim 1)
- Let $1_\beta \in \tilde{H}_{n-1}(S^{n-1}_\beta)$ be the generator.
- $d_{\alpha\beta} \cdot 1_\beta = (\deg \Delta_{\alpha\beta}) \cdot 1_\beta = q_{\beta *} q_* \varphi_{\alpha *} \partial(1)$
- $\partial^{\mathrm{CW}} e^n_\alpha = q_* \varphi_{\alpha *} \partial(1) = \sum_\beta q^{-1}_{\beta *} q_{\beta *} q_* \varphi_{\alpha *} \partial(1) = \sum_\beta q^{-1}_{\beta *}(d_{\alpha\beta} \cdot 1_\beta) = \sum_\beta d_{\alpha\beta} q^{-1}_{\beta *}(1_\beta) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$

□

Exercise 127 (HW6)

pg. 155-159 Problems 4, 5, 11, 12, 15, 17

Homology and the Fundamental Group

Theorem 128 (H_1 is the abelianization of π_1)

Define

$$h : \pi_1(X, x_0) \to H_1(X)$$

as follows. For a loop $f : I \to X$ *based at* x_0 *let* $\hat{f} : \Delta^1 \to X$ *be*

$$\hat{f}((1-t)e_0 + te_1) = f(t)$$

and set

$$h([f]_\pi) = [\hat{f}]_H$$

where $[f]_\pi$ *is the class of* f *in* $\pi_1(X, x_0)$ *and* $[\hat{f}]_H$ *is the class of* $\hat{f}$ *in* $H_1(X)$.

Then h *is a well defined group homomorphism and if* X *is path connected then* h *is surjective with kernel* $[\pi_1(X, x_0), \pi_1(X, x_0)]$.

Example 129

Let T^n be the n-torus $S^1 \times \cdots \times S^1$
Then

$$\pi_1(T^n, x_0) \cong \pi_1(S^1, x_0) \times \cdots \times \pi_1(S^1, x_0) \cong \mathbf{Z}^n$$

which is already abelian so

$$H_1(T^n) \cong \pi_1(T^n, x_0) \cong \mathbf{Z}^n$$

Example 130

Let Σ_g be the surface of genus g
Then

$$\pi_1(\Sigma_g) \cong \left\langle a_1, b_1, \cdots, a_g, b_g \middle| a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \right\rangle$$

Note that the one relation is in the commutator subgroup of the free group so

$$H_1(\Sigma_g) \cong \mathbf{Z}^{2g}$$

Example 131

Let R^n be the rose with n petals $S^1 \vee \cdots \vee S^1$
Then

$$\pi_1(R^n, x_0) \cong \pi_1(S^1, x_0) * \cdots * \pi_1(S^1, x_0) \cong F^n$$

is the free group on n generators so

$$H_1(R^n) \cong \mathbf{Z}^n$$

Proof of Theorem 128.

As above let h: $\pi_1(X, x_0) \to H_1(X)$ be defined on loops $f : I \to X$ based at x_0 by setting

$$h([f]_\pi) = [\hat{f}]_H$$

- Claim 1: $h([c_{x_0}]_\pi) = 0$
  - Let $\sigma : \Delta^2 \to X$ be the constant map $\sigma(p) = x_0$
  - Then

$$\begin{aligned}\partial\sigma &= \sigma \circ [e_1, e_2] - \sigma \circ [e_0, e_2] + \sigma \circ [e_0, e_1] \\ &= \hat{c}_{x_0} - \hat{c}_{x_0} + \hat{c}_{x_0} = \hat{c}_{x_0}\end{aligned}$$

  - So $h([c_{x_0}]_\pi) = [\hat{c}_{x_0}]_H = 0$

Proof of Theorem 128 *(continued)*.

- Claim 2: h is a well-defined function.
  - Suppose $f \simeq g$ as paths.
  - Then we have homotopy $F : I \times I \to X$ with $F(x,0) = f(x)$ and $F(x,1) = g(x)$
  - Let $v_0 = (0,0)$, $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (1,1)$,
  - Let
    $$\sigma_1 = F \circ [v_0, v_1, v_3]$$
    and
    $$\sigma_2 = F \circ [v_0, v_2, v_3]$$
  - Then
    $$\begin{aligned}\partial(\sigma_1 - \sigma_2) = F \circ [v_1, v_3] - F \circ [v_0, v_3] + F \circ [v_0, v_1] \\ - F \circ [v_2, v_3] + F \circ [v_0, v_3] - F \circ [v_0, v_2] \\ = \hat{c}_{f(1)} + \hat{f} - \hat{g} + -\hat{c}_{f(0)} \sim \hat{f} - \hat{g}\end{aligned}$$
  - Thus $\hat{f} \sim \hat{g}$ and if f and g are loops $[\hat{f}]_H = [\hat{g}]_H$ so h is well-defined.

Proof of Theorem 128 *(continued)*.

- Claim 3: $h([f \cdot g]_\pi) = [\hat{f}]_H + [\hat{g}]_H$
  - In fact we will show that $\widehat{f \cdot g} \sim \hat{f} + \hat{g}$ for any paths (with $f(1) = g(0)$).
  - Let $e_0, e_1, e_2 \in \mathbf{R}^3$ be the standard basis.
  - Let

$$\sigma(t_0 e_0 + t_1 e_1 + t_2 e_2) = \begin{cases} f\left(2t_2 + t_1\right), & t_2 + \frac{t_1}{2} \leq \frac{1}{2} \\ g\left(2t_2 + t_1 - 1\right), & t_2 + \frac{t_1}{2} \geq \frac{1}{2} \end{cases}$$

  - Then

$$\begin{aligned} \partial\sigma &= \sigma \circ [e_1, e_2] - \sigma \circ [e_0, e_2] + \sigma \circ [e_0, e_1] \\ &= \hat{g} - \widehat{f \cdot g} + \hat{f} \end{aligned}$$

  - Hence $\widehat{f \cdot g} \sim \hat{f} + \hat{g}$.
  - In particular if f and g are loops $h([f \cdot g]_\pi) = [\widehat{f \cdot g}]_H = [\hat{f}]_H + [\hat{g}]_H$.
- Therefore h is a homomorphism.
- $H_1(X)$ is abelian so $[\pi_1(X, x_0), \pi_1(X, x_0)] \subset \ker h$.

Proof of Theorem 128 *(continued)*.

- Note: By Claims 1 and 3 $-\hat{f} \sim \hat{\bar{f}}$ where $\bar{f}$ is the reverse of the path f.
- Claim 4: h is surjective if X is path-connected.
  - Let $\eta \in H_1(X)$ be a homology class represented by the 1-cycle $\sum_{i=1}^{n} k_i \sigma_i$.
  - WLOG we may assume $k_i = \pm 1$
  - Let $\lambda : I \to \Delta^1$ be the map $\lambda(t) = (1-t)e_0 + te_1$
  - Let
$$f_i = \begin{cases} \sigma_i \circ \lambda, & k_i = 1 \\ \overline{\sigma_i \circ \lambda}, & k_i = -1 \end{cases}$$
  - By the Note above
$$\sum_{i=1}^{n} \hat{f}_i \sim \sum_{i=1}^{n} k_i \sigma_i$$
  - If some f_j is not a loop then $\partial \hat{f}_j = f_j(1) - f_j(0) \nsim 0$. So there must be f_k with $f_k(0) = f_j(1)$
  - By Claim 3 $\widehat{f_j \cdot f_k} \sim \hat{f}_j + \hat{f}_k$
  - So left sum has fewer nonloops.
$$\widehat{f_j \cdot f_k} + \sum_{i \neq j,k} \hat{f}_i \sim \sum_{i=1}^{n} \hat{f}_i$$

Proof of Theorem 128 *(continued)*.

- Hence we may assume each f_i is a loop.
- X is path connected so let γ_i be a path from x_0 to the base point of f_i.
- By Claim 3 and the Note

$$\widehat{\gamma_i \cdot f_i \cdot \bar{\gamma}_i} \sim \hat{\gamma}_i + \hat{f}_i + \hat{\bar{\gamma}}_i \sim \hat{\gamma}_i + \hat{f}_i - \hat{\gamma}_i \sim \hat{f}_i$$

- Thus

$$\sum_{i=1}^{n} \widehat{\gamma_i \cdot f_i \cdot \bar{\gamma}_i} \sim \sum_{i=1}^{n} \hat{f}_i$$

- Each $\widehat{\gamma_i \cdot f_i \cdot \bar{\gamma}_i}$ is a loop based at x_0 so applying Claim 3 again we get

$$h([\gamma_1 \cdot f_1 \cdot \bar{\gamma}_1 \cdot \cdots \cdot \gamma_n \cdot f_n \cdot \bar{\gamma}_n]_\pi) = \left[\sum_{i=1}^{n} \hat{f}_i\right]_H = \eta$$

- Thus $h : \pi_1(X, x_0) \to H_1(X)$ is surjective.

Proof of Theorem 128 *(continued)*.

- Claim 4: $\ker h \subset [\pi_1(X, x_0), \pi_1(X, x_0)]$
  - Suppose f is a loop based at x_0 such that $[\hat{f}]_H = 0$.
  - Then $\hat{f}$ is a 1-boundary so there is a 2-chain $\sum_{i=1}^n k_i\sigma_i$ s.t.

$$\partial \sum_{i=1}^{n} k_i\sigma_i = \hat{f}$$

  - Consider the finite set

$$V = \{\sigma_i(e_0), \sigma_i(e_1), \sigma_i(e_2) | 1 \leq i \leq n\}$$

  - For each $p \in V$ fix a path γ_p from x_0 to p choosing the constant path c_{x_0} for $p = x_0$.
  - Modify each σ_i to get a new singular simplex ς_i which contains a shrunken version of σ_i near its center and travels along $\gamma_{\sigma_i(e_j)}$ near its jth corner.
  - Then

$$\partial \sum_{i=1}^{n} k_i\varsigma_i = \widehat{c_{x_0} \cdot f \cdot c_{x_0}}$$

  - Note that each face $\varsigma_i \circ [e_1, e_2] \circ \lambda$, $\varsigma_i \circ [e_0, e_2] \circ \lambda$, and $\varsigma_i \circ [e_0, e_1] \circ \lambda$ are loops based at x_0.

Proof of Theorem 128 *(continued)*.

- Let $L = \{\text{loops based at } x_0\}$
- By the universal property of the free abelian group there is a homomorphism

$$\rho : \mathbf{Z}[L] \to \pi_1(X, x_0)_{\mathrm{ab}}$$

  sending the loop l to its class $[l]$
- Note that $\mathbf{Z}[L] \subset C_1(X)$
- Already in $\mathbf{Z}[L]$ we have

$$\sum_{i=1}^{n} k_i(\varsigma_i \circ [e_1, e_2] \circ \lambda - \varsigma_i \circ [e_0, e_2] \circ \lambda + \varsigma_i \circ [e_0, e_1] \circ \lambda) = c_{x_0} \cdot f \cdot c_{x_0}$$

- So applying ρ to both sides we get equality in $\pi_1(X, x_0)_{\mathrm{ab}}$
- The map ς_i shows that in $\pi_1(X, x_0)$ the composition

$$(\varsigma_i \circ [e_1, e_2] \circ \lambda) \cdot (\varsigma_i \circ [e_0, e_1] \circ \lambda) \cdot \overline{(\varsigma_i \circ [e_0, e_2] \circ \lambda)}$$

  is nullhomotopic.

Proof of Theorem 128 *(continued)*.

- So

$$\begin{aligned} &\rho(\varsigma_i \circ [e_1, e_2] \circ \lambda - \varsigma_i \circ [e_0, e_2] \circ \lambda + \varsigma_i \circ [e_0, e_1] \circ \lambda) \\ &\quad = \rho\left((\varsigma_i \circ [e_1, e_2] \circ \lambda) \cdot (\varsigma_i \circ [e_0, e_1] \circ \lambda) \cdot \overline{(\varsigma_i \circ [e_0, e_2] \circ \lambda)} \right) \\ &\quad = \left[(\varsigma_i \circ [e_1, e_2] \circ \lambda) \cdot (\varsigma_i \circ [e_0, e_1] \circ \lambda) \cdot \overline{(\varsigma_i \circ [e_0, e_2] \circ \lambda)} \right] \\ &\quad = [c_{x_0}] = 0 \end{aligned}$$

- So

$$\begin{aligned} \rho(f) &= \rho(c_{x_0} f c_{x_0}) \\ &= \rho\left(\sum_{i=1}^{n} k_i (\varsigma_i \circ [e_1, e_2] \circ \lambda - \varsigma_i \circ [e_0, e_2] \circ \lambda + \varsigma_i \circ [e_0, e_1] \circ \lambda \right) \\ &= 0 \end{aligned}$$

- Thus $[f] \in [\pi_1(X, x_0), \pi_1(X, x_0)]$ □

Homology with coefficients

Definition 132 (Homology with coefficients)

For X a space and G an abelian group let

$$C_n(X;G) = \left\{ \sum_{i=1}^{k} g_i \sigma_i \middle| \sigma_i \text{ an } n\text{-simplex}, \ g_i \in G \right\}$$

For an n-simplex σ and $g \in G$ let $\partial : C_n(X;G) \to C_{n-1}(X;G)$ satisfy

$$\partial(g \cdot \sigma) = \sum_{i=1}^{k} (-1)^i g \partial^i \sigma$$

and extend linearly.

The n**th homology of** X **with coefficents in** G is

$$H_n(X;G) = H_n(C(X;G))$$