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## Definition 132 (Bilinear function)

A, B, C abelian groups

\[ \varphi : A \times B \rightarrow C \]

is **bilinear** if

\[ \varphi(a_1 + a_2, b) = \varphi(a_1, b) + \varphi(a_2, b) \]

and

\[ \varphi(a, b_1 + b_2) = \varphi(a, b_1) + \varphi(a, b_2) \]
Definition 133 (Tensor product of abelian groups)

$A, B$ abelian groups. The tensor product of $A$ and $B$ is the group

$$A \otimes B = \frac{\mathbb{Z}[A \times B]}{\langle (a, b_1 + b_2) - (a, b_1) - (a, b_2), (a_1 + a_2, b) - (a_1, b) - (a_2, b) \rangle}$$

The class of $(a, b)$ in $A \otimes B$ is written $a \otimes b$.

Note:

- $\{ (a, b) \}$ generates $\mathbb{Z}[A \times B]$
- so $\{ a \otimes b \}$ generates $A \otimes B$
- but just as general element of $\mathbb{Z}[A \times B]$ is of the form
  $$\sum_{i=1}^{k} n_i (a_i, b_i)$$
- general element of $A \otimes B$ is of the form
  $$\sum_{i=1}^{k} n_i \cdot a_i \otimes b_i$$
Lemma 134 (Universal property of the tensor product)

Let $A, B$ be abelian groups. Let

$$\kappa : A \times B \to A \otimes B$$

be the map $\kappa(a, b) = a \otimes b$.

- $\kappa$ is bilinear
- for any bilinear $\varphi : A \times B \to C$
- there exists a unique homomorphism $\varphi' : A \otimes B \to C$
- such that $\varphi = \varphi' \circ \kappa$.

\[
\begin{array}{ccc}
A \times B & \xrightarrow{\kappa} & A \otimes B \\
\downarrow \varphi & & \Downarrow \exists! \varphi' \\
C & & C
\end{array}
\]
Lemma 135 (Tensor product of homomorphisms)

Given homomorphisms of abelian groups

\[ f : A \to A' \]
\[ g : B \to B' \]

There is unique homomorphism

\[ f \otimes g : A \otimes B \to A' \otimes B' \]

satisfying

\[ (f \otimes g)(a \otimes b) = f(a) \otimes g(b) \]

Idea of proof.

Apply universal property to the map \( \varphi : A \times B \to A' \otimes B' \) given by

\[ \varphi(a, b) = f(a) \otimes g(b) \]
Proposition 136 (Properties of the tensor product)

1. \((f_2 \circ f_1) \otimes (g_2 \circ g_1) = (f_2 \otimes g_2) \circ (f_1 \otimes g_1)\)

That is, the following diagram commutes

\[
\begin{array}{ccc}
A_1 \otimes B_1 & \xrightarrow{f_1 \otimes g_1} & A_2 \otimes B_2 \xrightarrow{f_2 \otimes g_2} & A_3 \otimes B_3 \\
(f_2 \circ f_1) \otimes (g_2 \circ g_1)
\end{array}
\]

2. \((\bigoplus_\alpha A_\alpha) \otimes B \cong \bigoplus_\alpha (A_\alpha \otimes B)\)

3. \(A \otimes B \cong B \otimes A\) via the map \(a \otimes b \mapsto b \otimes a\)

4. \(\mathbb{Z} \otimes A \cong A\) via the map \(1 \otimes a \mapsto a\)

5. For sets \(S\) and \(T\)

\(\mathbb{Z}[S] \otimes \mathbb{Z}[T] \cong \mathbb{Z}[S \times T]\) via the map \(s \otimes t \mapsto (s, t)\)
Proposition 137

If \( A' \subset A \) and \( B' \subset B \) are abelian groups then

\[
(A/A') \otimes (B/B') \cong \frac{A \otimes B}{i_{A'} \otimes 1_B(A' \otimes B) + 1_A \otimes i_{B'}(A \otimes B')}
\]

Proof.

- Let
  
  \[
  q_1 : A \to A/A'
  \]
  
  \[
  q_2 : B \to B/B'
  \]
  
  be the quotient maps

- Then by Lemma 135 we have

  \[
  q_1 \otimes q_2 : A \otimes B \to (A/A') \otimes (B/B')
  \]

- Claim 1: \( A' \otimes B \subset \ker q_1 \otimes q_2 \). If \( a' \in A' \) and \( b \in B \) then

  \[
  q_1 \otimes q_2(a' \otimes b) = q_1(a') \otimes q_2(b) = 0 \otimes q_2(b) = 0
  \]
Proof of Proposition 137 (continued).

\[q_1 : A \rightarrow A/A'\]
\[q_2 : B \rightarrow B/B'\]

- Similarly \( A \otimes B' \subset \ker q_1 \otimes q_2 \).
- Let \( K = i_{A'} \otimes 1_B (A' \otimes B) + 1_A \otimes i_{B'} (A \otimes B') \).
- \( q_1 \otimes q_2 (K) = 0 \) so \( q_1 \otimes q_2 \) induces homomorphism

\[Q : (A \otimes B)/K \rightarrow (A/A') \otimes (B/B')\]

- Define

\[r : (A/A') \times (B/B') \rightarrow (A \otimes B)/K\]

by setting

\[r([a],[b]) = [a \otimes b]\]

- \( r \) is well-defined and bilinear.
Proof of Proposition 137 (continued).

\[ q_1 : A \rightarrow A/A' \]
\[ q_2 : B \rightarrow B/B' \]

- By universal property get homomorphism

\[ R : (A/A') \otimes (B/B') \rightarrow (A \otimes B)/K \]

with

\[ R([a] \otimes [b]) = [a \otimes b] \]

- \( QR = 1 \) and \( RQ = 1 \)
- Hence

\[ (A/A') \otimes (B/B') \cong \frac{A \otimes B}{i_A' \otimes 1_B (A' \otimes B) + 1_A \otimes i_{B'} (A \otimes B')} \]
Example 138

\[(\mathbb{Z}/n\mathbb{Z}) \otimes (\mathbb{Z}/m\mathbb{Z}) \cong \frac{\mathbb{Z} \otimes \mathbb{Z}}{n\mathbb{Z} \otimes \mathbb{Z} + \mathbb{Z} \otimes m\mathbb{Z}} \]

\[\cong \frac{\mathbb{Z}}{n\mathbb{Z} + m\mathbb{Z}} \cong \mathbb{Z} / \gcd(n, m)\mathbb{Z} \]

For example

\[(\mathbb{Z}/6\mathbb{Z}) \otimes (\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/6\mathbb{Z} \]

\[(\mathbb{Z}/15\mathbb{Z}) \otimes (\mathbb{Z}/12\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z} \]

\[(\mathbb{Z}/9\mathbb{Z}) \otimes (\mathbb{Z}/8\mathbb{Z}) \cong \mathbb{Z}/\mathbb{Z} = 0 \]
Right exactness of tensor product

Proposition 139 (Right exactness of tensor product)

If

\[ A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \]

is exact then for all \( D \)

\[ A \otimes D \xrightarrow{\alpha \otimes 1_D} B \otimes D \xrightarrow{\beta \otimes 1_D} C \otimes D \to 0 \]

is exact.

Proof of Proposition 139.

- Define

\[ \varphi : C \times D \to \frac{B \otimes D}{\alpha \otimes 1_D(A \otimes D)} \]

by setting \( \varphi(\beta(b), d) = [b \otimes d] \)

- Claim: \( \varphi \) is well-defined
Proof of Proposition 139 (continued).

- Suppose $\beta(b) = \beta(b')$
- Then $b - b' \in \ker \beta$ thus there is $a \in A$ with $\alpha(a) = b - b'$
- Thus

$$b \otimes d - b' \otimes d = (b - b') \otimes d = \alpha(a) \otimes d = \alpha \otimes 1_D(a \otimes d)$$

- so $\varphi$ is well-defined.
- Claim 2: $\varphi$ is bilinear

$$\varphi(\beta(b) + \beta(b'), d) = \varphi(\beta(b + b'), d) = [(b + b') \otimes d] = [b \otimes d + b' \otimes d] = [b \otimes d] + [b' \otimes d] = \varphi(\beta(b), d) + \varphi(\beta(b'), d)$$
Proof of Proposition 139 (continued).

\[ \varphi(\beta(b), d + d') = [b \otimes (d + d')] \]
\[ = [b \otimes d + b \otimes d'] \]
\[ = \varphi(\beta(b), d) + \varphi(\beta(b), d') \]

- Universal property of tensor product gives

\[ \Phi : C \otimes D \rightarrow \frac{B \otimes D}{\alpha \otimes 1_D(A \otimes D)} \]

- Claim 3: \( \beta \otimes 1_D(\alpha \otimes 1_D(A \otimes D)) = 0 \)

\[ \beta \otimes 1_D(\alpha \otimes 1_D(a \otimes d)) = (\beta \circ \alpha) \otimes (1_D \circ 1_D)(a \otimes d) \]
\[ = \beta \alpha(a) \otimes d \]
\[ = 0 \otimes d = 0 \]
Proof of Proposition 139 (continued).

- So
  \[ \beta \otimes 1_D : B \otimes D \to C \otimes D \]
  
  induces
  \[ \Theta : \frac{B \otimes D}{\alpha \otimes 1_D(A \otimes D)} \to C \otimes D \]

- \[ \Theta \Phi(\beta(b) \otimes d) = \Theta([b \otimes d]) = \beta \otimes d \]
- \[ \Phi \Theta([b \otimes d]) = \Phi(\beta(b) \otimes d) = [b \otimes d] \]
- So \( \Phi \) is an isomorphism establishing exactness at \( B \otimes D \).
- Surjectivity of \( \beta \) gives surjectivity of \( \beta \otimes 1_D \) establishing exactness at \( C \otimes D \).
Example 140 (Tensor product is not left exact)

- Consider the exact sequence

\[ 0 \rightarrow \mathbb{Z} \xrightarrow{\times 5} \mathbb{Z} \rightarrow \mathbb{Z}/5\mathbb{Z} \]

- Tensor with \( \mathbb{Z}/5\mathbb{Z} \) to get

\[ 0 \rightarrow \mathbb{Z} \otimes (\mathbb{Z}/5\mathbb{Z}) \xrightarrow{(\times 5) \otimes 1} \mathbb{Z} \otimes (\mathbb{Z}/5\mathbb{Z}) \rightarrow \mathbb{Z}/5\mathbb{Z} \otimes \mathbb{Z}/5\mathbb{Z} \]

- which becomes

\[ 0 \rightarrow \mathbb{Z}/5\mathbb{Z} \xrightarrow{0} \mathbb{Z}/5\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/5\mathbb{Z} \]

- No longer exact at leftmost \( \mathbb{Z}/5\mathbb{Z} \)
Tensor products and chain complexes

Given a chain complex \((C, \partial)\)

\[
\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \cdots
\]

and an abelian group \(G\) we get a new chain complex \((C \otimes G, \partial \otimes 1_G)\).

\[
\cdots \xrightarrow{\partial_{n+1} \otimes 1_G} C_n \otimes G \xrightarrow{\partial_n \otimes 1_G} C_{n-1} \otimes G \xrightarrow{\partial_{n-1} \otimes 1_G} C_{n-2} \otimes G \xrightarrow{\partial_{n-2} \otimes 1_G} \cdots
\]

Lemma 141

If \(C\) is a chain complex then \(C \otimes G\) is a chain complex.

Proof.

\[
(\partial_n \otimes 1_G) \circ (\partial_{n-1} \otimes 1_G) = (\partial_n \circ \partial_{n-1}) \otimes 1_G = 0 \otimes 1_G = 0
\]
Lemma 142

If

\[ f : C \rightarrow D \]

is a chain map then

\[ f \otimes 1_G : C \otimes G \rightarrow D \otimes G \]

is a chain map.

Proof.

\[ f : C \rightarrow D \] is a chain map so \( \partial f = f \partial \)

\[
(f \otimes 1_G) \circ (\partial \otimes 1_G) = (f \circ \partial) \otimes 1_G \\
= (\partial \circ f) \otimes 1_G \\
= (\partial \otimes 1_G) \circ (f \otimes 1_G)
\]
Lemma 143

If

\[ T : C \rightarrow D \]

is a chain homotopy between chain maps \( f, g : C \rightarrow D \) then

\[ T \otimes 1_G : C \otimes G \rightarrow D \otimes G \]

is a chain homotopy between chain maps \( f \otimes 1_G \) and \( g \otimes 1_G \)

Proof.

\( T : C \rightarrow D \) is a chain homotopy so \( \partial T + T \partial = g - f \)

\[
(\partial \otimes 1) \circ (T \otimes 1) + (T \otimes 1) \circ (\partial \otimes 1) = (\partial T) \otimes 1 + (T \partial) \otimes 1
\]

\[
= (\partial T + T \partial) \otimes 1
\]

\[
= (g - f) \otimes 1
\]

\[
= g \otimes 1 - f \otimes 1
\]
**Definition 144 (Homology with coefficients in $G$)**

Given a chain complex $C$ let

$$H_n(C; G) = H_n(C \otimes G)$$

If $X$ is a space let

$$H_n(X; G) = H_n(C(X) \otimes G)$$

$$\tilde{H}_n(X; G) = H_n(\tilde{C}(X) \otimes G)$$

$$H_n^{CW}(X; G) = H_n(C^{CW}(X) \otimes G)$$

For $(X, A)$ a topological pair let

$$H_n(X, A; G) = H_n(C(X, A) \otimes G)$$
Example 145 (Homology of $\mathbb{R}P^4$)

Standard CW chain complex for $\mathbb{R}P^4$ is

$$
\cdots \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 0} 0
$$

Tensoring with $\mathbb{Z}_2$ we get $C_{\text{CW}}(\mathbb{R}P^4) \otimes \mathbb{Z}_2$

$$
\cdots \rightarrow 0 \rightarrow \mathbb{Z}_2 \xrightarrow{\times 0} \mathbb{Z}_2 \xrightarrow{\times 0} \mathbb{Z}_2 \xrightarrow{\times 0} \mathbb{Z}_2 \xrightarrow{\times 0} 0
$$

So

$$H_n^{\text{CW}}(\mathbb{R}P^4; \mathbb{Z}_2) \cong \begin{cases} 
\mathbb{Z}_2, & 0 \leq n \leq 4 \\
0, & \text{otherwise}
\end{cases}$$

Tensoring with $\mathbb{Q}$ we get $C_{\text{CW}}(\mathbb{R}P^4) \otimes \mathbb{Q}$

$$
\cdots \rightarrow 0 \rightarrow \mathbb{Q} \xrightarrow{\times 2} \mathbb{Q} \xrightarrow{\times 0} \mathbb{Q} \xrightarrow{\times 2} \mathbb{Q} \xrightarrow{\times 0} \mathbb{Q} \xrightarrow{\times 0} 0
$$

So

$$H_n^{\text{CW}}(\mathbb{R}P^4; \mathbb{Q}) \cong \begin{cases} 
\mathbb{Q}, & n = 0 \\
0, & n \neq 0
\end{cases}$$
Note that $C \otimes \mathbb{Z} = C$ so

$$H_n(C; \mathbb{Z}) \cong H_n(C)$$

Why consider homology with other coefficient groups?

1. If $G$ is a right $R$-module then $H_n(C; G)$ is a right $R$-module.
2. In particular if $G$ is a field then $H_n(C; G)$ is a vector space over $G$.
3. Computation of $H_n(C; \mathbb{Z}_2)$ are often easier.

What advantages does the coefficient group $\mathbb{Z}$ have?

1. Homology groups with coefficients in $\mathbb{Z}$ are the strongest topological invariants.
Homology with coefficients in $\mathbb{Z}$ are “universal” since they determine homology with coefficients in $G$ for all abelian groups $G$.

**Theorem 146 (Universal coefficient theorem)**

*For $C$ a chain complex there is a split short exact sequence*

$$0 \rightarrow H_n(C) \otimes G \rightarrow H_n(C; G) \rightarrow \text{Tor}(H_{n-1}(C), G) \rightarrow 0.$$  

*This sequence is natural in that if $C \rightarrow D$ is a chain map then we get commutative*

$$
\begin{array}{cccccc}
0 & \rightarrow & H_n(C) \otimes G & \rightarrow & H_n(C; G) & \rightarrow & \text{Tor}(H_{n-1}(C), G) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H_n(D) \otimes G & \rightarrow & H_n(D; G) & \rightarrow & \text{Tor}(H_{n-1}(D), G) & \rightarrow & 0
\end{array}
$$
The Tor functor

**Definition 147 (Free resolution)**

A free resolution of the abelian group $A$ is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to A \to 0$$

where each $F_i$ is free abelian.

**Example 148 (Two free resolutions of $\mathbb{Z}_5$)**

\[
\begin{align*}
\cdots & \to 0 \to 0 \to \mathbb{Z} \xrightarrow{\times 5} \mathbb{Z} \to \mathbb{Z}_5 \to 0 \\
\cdots & \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 5} \mathbb{Z} \to \mathbb{Z}_5 \to 0
\end{align*}
\]
Theorem 149 (Existence of free resolutions)

Every abelian group $A$ has a free resolution

$$
\cdots \to 0 \to R \to F \to A \to 0
$$

where $F$ and $R$ are free abelian.

Theorem 150

Every subgroup of a free abelian group is a free abelian group.

Proof of Theorem 149.

We have a surjection

$$
\mathbb{Z}[A] \xrightarrow{p} A
$$

given by $p(a) = a$. Let $F = \mathbb{Z}[A]$ and $R = \ker p$.

Then

$$
\cdots \to 0 \to R \to F \xrightarrow{p} A \to 0
$$

is a free resolution of $A$.  

Definition 151 (The Tor functor)

Let $A$ and $B$ be abelian groups and

$$
\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \rightarrow A \rightarrow 0
$$

be a free resolution of $A$.

Let $\partial_0 : F_0 \rightarrow 0$ be the 0 map so we have a chain complex $(F, \partial)$

$$
\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0
$$

Tensoring with $B$ we get the chain complex $F \otimes B$

$$
\cdots \xrightarrow{\partial_3 \otimes 1_B} F_2 \otimes B \xrightarrow{\partial_2 \otimes 1_B} F_1 \otimes B \xrightarrow{\partial_1 \otimes 1_B} F_0 \otimes B \xrightarrow{\partial_0 \otimes 1_B} 0
$$

$$
\text{Tor}_n(A, B) = H_n(F \otimes B)
$$
Theorem 152 (Tor is well-defined)

Tor_n(A, B) is independent of the free resolution F of A.
Lemma 153 (Free abelian groups are projective)

Suppose we have commutative

\[
\begin{array}{ccc}
F & \xrightarrow{\exists \psi} & 0 \\
 M' & \xrightarrow{i} & M & \xrightarrow{j} & M'' \\
\end{array}
\]

1. If \( F \) is a free abelian group
2. \( M' \xrightarrow{i} M \xrightarrow{j} M'' \) is exact

There is a homomorphism \( \psi \) making the diagram commute.

Proof.

- Let \( \{ e_\alpha \} \) be a free basis for \( F \)
- \( j \varphi(e_\alpha) = 0(e_\alpha) = 0 \) so \( \varphi(e_\alpha) \in \ker j = \text{Im} i \)
- Thus there is \( m'_\alpha \in M' \) such that \( i(m'_\alpha) = \varphi(e_\alpha) \)
- Let \( \psi(e_\alpha) = m'_\alpha \)
We will prove Theorem 152 using the following lemma

**Lemma 154**

*If $F$ and $F'$ are two free resolutions of $A$ then there is a chain map $f : F \to F'$ which is a chain homotopy equivalence.*

**Proof.**

- We have

$$
\cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \xrightarrow{0} 0
$$

$$
\cdots \xrightarrow{\partial_2'} F'_1 \xrightarrow{\partial_1'} F'_0 \xrightarrow{\varepsilon'} A \xrightarrow{0} 0
$$

- Apply Lemma 153 (free ab. grps. are proj.) with $\varphi = 1_A \varepsilon$
- Get $\psi : F_0 \to F'_0$ with $\varepsilon' \psi = 1_A \varepsilon$
- Let $f_0 = \psi$
- Now apply Lemma 153 with $\varphi = f_0 \partial_1$ to get $f_1 : F_1 \to F'_1$
- Continue inductively to get chain map $f : F \to F'$
Proof of Lemma 154 (continued).

- Now we will show that the chain homotopy type of \( f \) is unique.
- That is if \( g : F \to F' \) is another chain map extending \( 1_A : A \to A \) then there is a chain homotopy \( T \) between \( f \) and \( g \).
- Let \( \tau = g - f \).
- Assume inductively that \( \partial'_{n+1} T_n + T_{n-1} \partial_n = \tau_n \).
- We have (non-commutating diagram)

\[
\begin{array}{cccc}
F_{n+1} & \xrightarrow{\partial_{n+1}} & F_n & \xrightarrow{\partial_n} & F_{n-1} \\
\downarrow \tau_{n+1} & & \downarrow T_n & & \downarrow T_{n-1} \\
F'_{n+2} & \xrightarrow{\partial'_{n+2}} & F'_{n+1} & \xrightarrow{\partial'_{n+1}} & F'_{n} \\
\end{array}
\]

- Consider the map \( (\tau_{n+1} - T_n \partial_{n+1}) : F_{n+1} \to F'_{n+1} \)

\[
\partial'_{n+1}(\tau_{n+1} - T_n \partial_{n+1}) = \partial'_{n+1} \tau_{n+1} - \partial'_{n+1} T_n \partial_{n+1} \\
= \partial'_{n+1} \tau_{n+1} - (\tau_n - T_{n-1} \partial_n) \partial_{n+1} \\
= \partial'_{n+1} \tau_{n+1} - \tau_n \partial_{n+1} + T_{n-1} \partial_n \partial_{n+1} \\
= 0
\]
Proof of Lemma 154 (continued).

- Apply Lemma 153 (free ab. grps. are proj.) with
  \[ \varphi = \tau_{n+1} - T_n \partial_{n+1} \]
- Let \( T_{n+1} = \psi : F_{n+1} \to F'_{n+2} \)
- Then \( \partial'_{n+2} T_{n+1} = \varphi = \tau_{n+1} - T_n \partial_{n+1} \)
- Hence
  \[ \partial'_{n+2} T_{n+1} + T_n \partial_{n+1} = \tau_{n+1} = g_{n+1} - f_{n+1} \]
- We may start the induction in degree \(-2\) where all groups are 0.
- Thus the chain homotopy class of \( f : F \to F' \) is unique.
- Similarly we get \( f' : F' \to F \)
- \( 1_F : F \to F \) extends \( 1_A : A \to A \)
- so \( f' \circ f \) is chain homotopic to \( 1_F \)
- That is, \( f \) is a chain homotopy equivalence.
Now we finish proof that $\text{Tor}_n(A, B)$ is independent of free resolution of $A$.

**Proof of Theorem 152 (Tor is well-defined).**

- Let $F$ and $F'$ be two free resolutions of $A$
- By Lemma 154 we have a chain homotopy equivalence
  \[ f : F \to F' \]
- Hence there is a chain map $f' : F' \to F$ such that $f' \circ f$ is chain homotopic to $1_F$
- Tensoring with $B$ we get $(f' \otimes 1_B) \circ (f \otimes 1_B)$ is chain homotopic to $1_F \otimes B$
- Thus
  \[ (f \otimes 1_B)_* : H_n(F \otimes B) \to H_n(F' \otimes B) \]
  is an isomorphism
- $\text{Tor}_n(A, B) = H_n(F \otimes B) \cong H_n(F' \otimes B)$ is well-defined
• As we saw in Lemma 149 every abelian group $A$ has a free resolution

$$\cdots \to 0 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \to 0$$

• So we have the chain complexes $F$

$$\cdots \to 0 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0$$

• and $F \otimes B$

$$\cdots \to 0 \xrightarrow{\partial_2 \otimes 1_B} F_1 \otimes B \xrightarrow{\partial_1 \otimes 1_B} F_0 \otimes B \xrightarrow{\partial_0 \otimes 1_B} 0$$

•

$$\text{Tor}_n(A, B) = H_n(F \otimes B) = \begin{cases} 
\frac{F_0 \otimes B}{\text{Im}(\partial_1 \otimes 1_B)}, & n = 0 \\
\ker(\partial_1 \otimes 1_B), & n = 1 \\
0, & n \neq 0, 1
\end{cases}$$

• We can say more about $\text{Tor}_0(A, B)$. 

• We have the exact sequence

\[ F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \rightarrow 0 \]

• Which remains exact after tensoring with \( B \) so we get exact

\[ F_1 \otimes B \xrightarrow{\partial_1 \otimes 1_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes 1_B} A \otimes B \rightarrow 0 \]

• Hence

\[ \text{Tor}_0(A, B) \cong \frac{F_0 \otimes B}{\text{Im}(\partial_1 \otimes 1_B)} \cong A \otimes B \]

• Since \( \text{Tor}_1(A, B) \) is the only new object we define

**Definition 155**

\[ \text{Tor}(A, B) = \text{Tor}_1(A, B) \]
• Note that if we have exact

$$0 \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \to 0$$

• Then we have exact

$$F_1 \otimes B \xrightarrow{\partial_1 \otimes 1_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes 1_B} A \otimes B \to 0$$

• and hence exact

$$0 \to \ker(\partial_1 \otimes 1_B) \to F_1 \otimes B \xrightarrow{\partial_1 \otimes 1_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes 1_B} A \otimes B \to 0$$

• and hence exact

$$0 \to \text{Tor}(A, B) \to F_1 \otimes B \xrightarrow{\partial_1 \otimes 1_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes 1_B} A \otimes B \to 0$$

• In particular if \(\text{Tor}(A, B) = 0\) then tensoring with \(B\) preserves exactness.
Example 156

Let's compute $\text{Tor}(\mathbb{Z}_{60}, \mathbb{Z}_{42})$

- Free resolution $F$ of $\mathbb{Z}_{60}$

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\times 60} \mathbb{Z} \to \mathbb{Z}_{60} \to 0$$

- $F \otimes \mathbb{Z}_{42}$ is

$$0 \to \mathbb{Z} \otimes \mathbb{Z}_{42} \xrightarrow{(\times 60) \otimes 1} \mathbb{Z} \otimes \mathbb{Z}_{42} \to 0$$

- Simplifying

$$0 \to \mathbb{Z}_{42} \xrightarrow{\times 60} \mathbb{Z}_{42} \to 0$$

- Hence

$$\text{Tor}(\mathbb{Z}_{60}, \mathbb{Z}_{42}) \cong \ker(\times 60) \cong \frac{\mathbb{Z}}{42\mathbb{Z} + 60\mathbb{Z}} \cong \frac{\mathbb{Z}}{\gcd(42, 60)\mathbb{Z}} \cong \mathbb{Z}_{6}$$
Proposition 157 (Properties of Tor)

1. \( \text{Tor}(A, B) \cong \text{Tor}(B, A) \)
2. \( \text{Tor}(\bigoplus_{\alpha} A_{\alpha}, B) \cong \bigoplus_{\alpha} \text{Tor}(A_{\alpha}, B) \)
3. \( \text{Tor}(A, B) = 0 \) if \( A \) or \( B \) is free or torsion free.
4. \( \text{Tor}(A, B) \cong \text{Tor}(A_{\text{tor}}, B) \) where \( A_{\text{tor}} \) is the torsion subgroup of \( A \).
5. \( \text{Tor}(\mathbb{Z}_n, A) \cong \ker \left( A \xrightarrow{\times n} A \right) \)
6. The short exact sequence

\[
0 \to B \to C \to D \to 0
\]

yields a natural exact sequence

\[
0 \to \text{Tor}(A, B) \to \text{Tor}(A, C) \to \text{Tor}(A, D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0
\]
Proof of Proposition 157.

2. \( \text{Tor}(\bigoplus_{\alpha} A_{\alpha}, B) \cong \bigoplus_{\alpha} \text{Tor}(A_{\alpha}, B) \)
   
   - Let \( F_{\alpha} \) be a free resolution of \( A_{\alpha} \)
   - Then \( \bigoplus_{\alpha} F_{\alpha} \) is a free resolution of \( A_{\alpha} \)
   
   \[
   \text{Tor}(\bigoplus_{\alpha} A_{\alpha}, B) \cong H_1((\bigoplus_{\alpha} F_{\alpha}) \otimes B)
   \cong H_1(\bigoplus_{\alpha} (F_{\alpha} \otimes B))
   \cong \bigoplus_{\alpha} H_1(F_{\alpha} \otimes B)
   \cong \bigoplus_{\alpha} \text{Tor}(A_{\alpha}, B)
   \]

5. \( \text{Tor}(\mathbb{Z}_n, A) \cong \ker \left( A \xrightarrow{\times n} A \right) \)
   
   - Use the free resolution of \( \mathbb{Z}_n \)
   
   \[
   \cdots \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \rightarrow A \rightarrow 0
   \]
   
   - Tensoring with \( A \) simplifies to
   
   \[
   \cdots \rightarrow 0 \rightarrow A \xrightarrow{(\times n)} A \rightarrow 0
   \]
Proof of Proposition 157 (continued).

3 Tor($A$, $B$) = 0 if $A$ or $B$ is free (we will address torsion free later)
- Suppose $A$ is free.
- Use the free resolution of $A$

\[ \cdots \to 0 \to 0 \to A \to A \to 0 \]

- Tor($A$, $B$) = ker($0 \to 0$) = 0
- Suppose $B$ is $\mathbb{Z}$
- Then tensoring an exact free resolution $0 \to F_1 \to F_0 \to A \to 0$ with $B$ remains exact
- Suppose $B \cong \bigoplus \alpha \mathbb{Z}$
- Then tensoring an exact free resolution $0 \to F_1 \to F_0 \to A \to 0$ with $B$ is a direct sum of exact sequences which is exact.
Proof of Proposition 157 (continued).

The short exact sequence

\[ 0 \to B \to C \to D \to 0 \]

yields a natural exact sequence

\[ 0 \to \text{Tor}(A, B) \to \text{Tor}(A, C) \to \text{Tor}(A, D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0 \]

- Choose a free resolution \( F \) of the form \( 0 \to F_1 \to F_0 \to A \to 0 \)
- All the terms of \( F \) are free so tensoring \( 0 \to B \to C \to D \to 0 \) with \( F_n \) remains exact.
- Get short exact sequence of chain complexes

\[ 0 \to (F \otimes B) \to (F \otimes C) \to (F \otimes D) \to 0 \]

- Apply Snake Lemma to get natural exact sequence above.
Proof of Proposition 157 (continued).

1. \( \text{Tor}(A, B) \cong \text{Tor}(B, A) \)
   - Consider the six term exact sequence from part 6 coming from the short exact
     \[ 0 \to F_1 \to F_0 \to B \to 0 \]
   - \( F_0 \) and \( F_1 \) are free so by part 3, \( \text{Tor}(A, F_1) \cong \text{Tor}(A, F_0) \cong 0 \)
   - 
     \[ 0 \to \text{Tor}(A, B) \to A \otimes F_1 \to A \otimes F_0 \to A \otimes B \to 0 \]
Proof of Proposition 157 (continued).

1. \( \text{Tor}(A, B) \cong \text{Tor}(B, A) \) (continued)

- We will define a homomorphism \( \gamma : \text{Tor}(A, B) \to \text{Tor}(B, A) \) preserving commutativity
- Let \( x \in \text{Tor}(A, B) \)
- Claim: \( \tau \alpha(x) \in \text{Im} \alpha' \)
- By commutativity \( \beta' \tau \alpha(x) = \mu \beta \alpha(x) = \mu(0) = 0 \)
- So \( \tau \alpha(x) \in \ker \beta' = \text{Im} \alpha' \)
- By injectivity of \( \alpha' \) there is a unique \( x' \in \text{Tor}(B, A) \) with \( \alpha'(x') = \tau \alpha(x) \)
- Set \( \gamma(x) = x' \).
- \( \gamma \) takes 0 to 0 and sums to sums so it is a homomorphism.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Tor}(A, B) & \xrightarrow{\alpha} & A \otimes F_1 & \xrightarrow{\beta} & A \otimes F_0 & \longrightarrow & A \otimes B & \longrightarrow & 0 \\
& & \downarrow{\cong} & \downarrow{\cong} & \downarrow{\gamma} & \downarrow{\tau} & \downarrow{\cong} & \downarrow{\cong} & \downarrow{\cong} & \downarrow{\cong} & \\
0 & \longrightarrow & \text{Tor}(B, A) & \xrightarrow{\alpha'} & F_1 \otimes A & \xrightarrow{\beta'} & F_0 \otimes A & \longrightarrow & B \otimes A & \longrightarrow & 0 \\
\end{array}
\]
Proof of Proposition 157 (continued).

1 Tor(A, B) \cong Tor(B, A) (continued)

- Add some trivial groups and homomorphisms to get

\[
\begin{array}{c}
0 \longrightarrow 0 \longrightarrow \text{Tor}(A, B) \longrightarrow A \otimes F_1 \longrightarrow A \otimes F_0 \\
\downarrow \cong \downarrow \cong \downarrow \gamma \quad \tau \downarrow \cong \mu \downarrow \cong \\
0 \longrightarrow 0 \longrightarrow \text{Tor}(B, A) \longrightarrow F_1 \otimes A \longrightarrow F_0 \otimes A
\end{array}
\]

- Now apply Five Lemma to show \( \gamma : \text{Tor}(A, B) \to \text{Tor}(B, A) \) is an isomorphism.
Proof of Proposition 157 (continued).

3 Tor$(A, B) \cong 0$ if $A$ or $B$ is torsion free.

- Applying part 1 assume $B$ is torsion free.
- Let

$$0 \to F_1 \xrightarrow{\partial_1} F_0 \to A$$

be a free resolution of $A$.
- Get exact

$$0 \to \text{Tor}(A, B) \to F_1 \otimes B \xrightarrow{\partial_1 \otimes 1_B} F_0 \otimes B$$

- Claim $\partial_1 \otimes 1_B$ is injective.
- Suppose $\sum_i f_i \otimes b_i \in \ker \partial_1 \otimes 1_B$
- Then then $\sum_i (\partial_1 f_i) \otimes b_i = 0$ in $F_0 \otimes B$.
- Hence in $\mathbb{Z}[F_0 \times B]$

$$\sum_i (\partial_1 f_i, b_i) = \sum_j (f_j^0, b_j^1 + b_j^2) - (f_j^0, b_j^1) - (f_j^0, b_j^2)$$

$$+ \sum_k (f_k^1 + f_k^2, b_k^0) - (f_k^1, b_k^0) - (f_k^2, b_k^2)$$

- Let $B_0 \subset B$ be the subgroup generated by the finite set

$\{b_i, b_k^0, b_j^1, b_j^2\}$
Proof of Proposition 157 (continued).

3 Tor(A, B) ≅ 0 if A or B is torsion free. (continued)

• Hence in \( \mathbb{Z}[F_0 \times B_0] \)

\[
\sum_i (\partial_1 f_i, b_i) = \sum_j (f_j^0, b_j^1 + b_j^2) - (f_j^0, b_j^1) - (f_j^0, b_j^2)
\]

\[
+ \sum_k (f_k^1 + f_k^2, b_k^0) - (f_k^1, b_k^0) - (f_k^2, b_k^2)
\]

• Therefore in \( F_0 \otimes B_0 \)

\[
\sum_i (\partial_1 f_i) \otimes b_i = 0
\]

• Let \( B_0 \subset B \) is torsion free and finitely generated so free abelian.
• Hence

\[
F_1 \otimes B_0 \xrightarrow{\partial_1 \otimes 1_{B_0}} F_0 \otimes B_0
\]

is injective.
• Thus in \( F_1 \otimes B_0 \)

\[
\sum_i f_i \otimes b_i = 0
\]
Proof of Proposition 157 (continued).

3. Tor(A, B) ≅ 0 if A or B is torsion free. (continued)
   • Hence in $\mathbb{Z}[F_1 \times B_0]$

   $\sum_i (f_i, b_i) = \sum_n (f_n^3, b_n^4 + b_n^5) - (f_n^3, b_n^4) - (f_n^3, b_n^5)$
   
   $+ \sum_k (f_k^4 + f_k^5, b_k^3) - (f_k^4, b_k^3) - (f_k^4, b_k^3)$

   • This equality holds in $\mathbb{Z}[F_1 \times B]$
   • Thus in $F_1 \otimes B$

   $\sum_i f_i \otimes b_i = 0$

   • It follows that Tor(A, B) = ker $(\partial_1 \otimes 1_B) = 0$
Proof of Proposition 157 (continued).

4 Tor(A, B) ≃ Tor(A_{tor}, B) where A_{tor} is the torsion subgroup of A.

- We have the exact sequence 0 \to A_{tor} \to A \to (A/A_{tor}) \to 0
- Apply part 6 to get exact

\[ 0 \to \text{Tor}(B, A_{tor}) \to \text{Tor}(B, A) \to \text{Tor}(B, A/A_{tor}) \to \text{Tor}(B, A) \to \text{Tor}(B, A) \to 0 \]

- A/A_{tor} is torsion free so by part 3 Tor(B, A/A_{tor}) = 0
- Get exact

\[ 0 \to \text{Tor}(B, A_{tor}) \to \text{Tor}(B, A) \to 0 \]

- Thus Tor(B, A_{tor}) ≃ Tor(B, A)
- and by part 1 Tor(A_{tor}, B) ≃ Tor(A, B)
Theorem 158 (Universal coefficient theorem)

If $C$ a chain complex with each $C_n$ free abelian there is a split short exact sequence

$$0 \to H_n(C) \otimes G \to H_n(C; G) \to \text{Tor}(H_{n-1}(C), G) \to 0.$$  

In particular

$$H_n(C; G) \cong (H_n(C) \otimes G) \oplus \text{Tor}(H_{n-1}(C), G)$$

The short exact sequence is natural in that if $C \to D$ is a chain map then we get commutative

$$
\begin{array}{cccccc}
0 & \to & H_n(C) \otimes G & \to & H_n(C; G) & \to \text{Tor}(H_{n-1}(C), G) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H_n(D) \otimes G & \to & H_n(D; G) & \to \text{Tor}(H_{n-1}(D), G) & \to 0 \\
\end{array}
$$
Proof of Theorem 158.

- Let $C$ be a chain complex with each $C_n$ free abelian.
- Let $Z_n = \ker \partial_n \subset C_n$ be the $n$-cycles.
- Let $B_n = \text{Im} \partial_{n+1} \subset C_n$ be the $n$-boundaries.
- View $Z$ and $B$ as chain complexes.
- $Z_n$ and $B_n$ are subgroups of free abelian groups and hence free abelian.
- Thus we have exact

$$0 \to Z_n \to C_n \to B_{n-1} \to 0$$

- View above as a free resolution of $B_{n-1}$.
- Get exact

$$0 \to \text{Tor}(B_{n-1}, G) \to Z_n \otimes G \to C_n \otimes G \to B_{n-1} \otimes G \to 0$$

- $B_{n-1}$ is free so $\text{Tor}(B_{n-1}, G) = 0$.
Proof of Theorem 158 (continued).

- $B_{n-1}$ is free so $\text{Tor}(B_{n-1}, G) = 0$
- Thus we have exact

$$0 \to Z_n \otimes G \to C_n \otimes G \to B_{n-1} \otimes G \to 0$$
- And hence exact sequence of chain complexes

$$0 \to Z \otimes G \to C \otimes G \to B \otimes G \to 0$$
- Apply Snake Lemma and fact that $\partial = 0$ for $Z$ and $B$ to get exact

$$\cdots \to B_n \otimes G \xrightarrow{\kappa} Z_n \otimes G \to H_n(C \otimes G) \to B_{n-1} \otimes G \xrightarrow{\kappa} Z_{n-1} \otimes G \to \cdots$$

where $\kappa$ is the connecting homomorphism
Proof of Theorem 158 (continued).

- What is $\kappa$?
- We have

\[
\begin{array}{ccccccc}
0 & \rightarrow & Z_n \otimes G & \rightarrow & C_n \otimes G & \stackrel{\partial_n \otimes 1}{\rightarrow} & B_{n-1} \otimes G & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & Z_{n-1} \otimes G & \rightarrow & C_{n-1} \otimes G & \stackrel{\partial_{n-1} \otimes 1}{\rightarrow} & B_{n-2} \otimes G & \rightarrow & 0
\end{array}
\]

- Given $b_{n-1} \otimes g \in B_{n-1} \otimes G$
- There is $c_n \in C_n$ such that $\partial_n c_n = b_{n-1}$
- Maps to $b_{n-1} \otimes g$ now viewed as a element of $C_{n-1} \otimes G$
- Which is the image of the element $b_{n-1} \otimes g \in Z_{n-1} \otimes G$
- Thus the connecting homomorphism is $\kappa = i_{n-1} \otimes 1$ where

\[i_{n-1} : B_{n-1} \rightarrow Z_{n-1}\]

is inclusion
Proof of Theorem 158 (continued).

- From long exact sequence

\[ \cdots \to B_n \otimes G \xrightarrow{i_n \otimes 1} Z_n \otimes G \to H_n(C \otimes G) \]
\[ \to B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes 1} Z_{n-1} \otimes G \to \cdots \]

- Get short exact sequence

\[ 0 \to \frac{Z_n \otimes G}{\text{Im}(i_n \otimes 1)} \to H_n(C \otimes G) \to \ker(i_{n-1} \otimes 1) \to 0 \]

- By the definition of $H_{n-1}(C)$ we have a free resolution

\[ 0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0 \]

- Tensoring with $G$ we get exact

\[ 0 \to \text{Tor}(H_n(C), G) \to B_n \otimes G \xrightarrow{i_n \otimes 1} Z_n \otimes G \]
\[ \to H_n(C) \otimes G \to 0 \]
Proof of Theorem 158 (continued).

- Apply exactness of

\[ 0 \to \text{Tor}(H_n(C), G) \to B_n \otimes G \xrightarrow{i_n \otimes 1} Z_n \otimes G \to H_n(C) \otimes G \to 0 \]

- To short exact sequence

\[ 0 \to \frac{Z_n \otimes G}{\text{Im}(i_n \otimes 1)} \to H_n(C \otimes G) \to \ker(i_{n-1} \otimes 1) \to 0 \]

- We see that

\[ \frac{Z_n \otimes G}{\text{Im}(i_n \otimes 1)} \cong H_n(C) \otimes G \]

- and

\[ \ker(i_{n-1} \otimes 1) \cong \text{Tor}(H_n(C), G) \]

- Conclude with exactness of

\[ 0 \to H_n(C) \otimes G \to H_n(C; G) \to \text{Tor}(H_{n-1}(C), G) \to 0 \]
Exercise 159 (HW7)

- *Hatcher pg. 155-159 Problems 19, 24, 40*
- *Hatcher pg. 267 Problems 1, 2*