Math 757
Homology theory

February 21, 2011

Review of tensor products

Definition 132 (Bilinear function)

A, B, C abelian groups

$$\varphi : A \times B \to C$$

is **bilinear** if

$$\varphi(a_1 + a_2, b) = \varphi(a_1, b) + \varphi(a_2, b)$$

and

$$\varphi(a, b_1 + b_2) = \varphi(a, b_1) + \varphi(a, b_2)$$

Definition 133 (Tensor product of abelian groups)

A, B abelian groups. The **tensor product** of A and B is the group

$$A \otimes B = \frac{\mathbf{Z}[A \times B]}{\left\langle \begin{array}{c} (a, b_1 + b_2) - (a, b_1) - (a, b_2) \\ (a_1 + a_2, b) - (a_1, b) - (a_2, b) \end{array} \right\rangle}$$

The class of (a, b) in $A \otimes B$ is written $a \otimes b$.

Note:

- $\{(a, b)\}$ generates $\mathbf{Z}[A \times B]$
- so $\{a \otimes b\}$ generates $A \otimes B$
- but just as general element of $\mathbf{Z}[A \times B]$ is of the form

$$\sum_{i=1}^{k} n_i (a_i, b_i)$$

- general element of $A \otimes B$ is of the form

$$\sum_{i=1}^{k} n_i \cdot a_i \otimes b_i$$

Lemma 134 (Universal property of the tensor product)

A, B abelian groups. Let

$$\kappa : A \times B \to A \otimes B$$

be the map $\kappa(a, b) = a \otimes b$.

- *Then κ is bilinear*
- *for any* bilinear $\varphi : A \times B \to C$
- *there exists a unique* homomorphism $\varphi' : A \otimes B \to C$
- *such that $\varphi = \varphi' \circ \kappa$.*

$$\begin{tikzcd}
A \times B \arrow[rr, "\varphi"] \arrow[dr, "\kappa"'] & & C \\
& A \otimes B \arrow[ur, dotted, "\exists! \varphi'"'] &
\end{tikzcd}$$

Lemma 135 (Tensor product of homomorphisms)

Given homomorphisms of abelian groups

$$f : A \to A'$$

$$g : B \to B'$$

There is unique homomorphism

$$f \otimes g : A \otimes B \to A' \otimes B'$$

satisfying

$$(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$$

Idea of proof.

Apply universal property to the map $\varphi : A \times B \to A' \otimes B'$ given by $\varphi(a, b) = f(a) \otimes g(b)$ $\square$

Proposition 136 (Properties of the tensor product)

1. $$(f_2 \circ f_1) \otimes (g_2 \circ g_1) = (f_2 \otimes g_2) \circ (f_1 \otimes g_1)$$

   That is, the following diagram commutes

   $$\begin{tikzcd} A_1 \otimes B_1 \arrow[r, "f_1 \otimes g_1"] \arrow[rr, bend right, "(f_2 \circ f_1) \otimes (g_2 \circ g_1)"'] & A_2 \otimes B_2 \arrow[r, "f_2 \otimes g_2"] & A_3 \otimes B_3 \end{tikzcd}$$

2. $$(\oplus_\alpha A_\alpha) \otimes B \cong \oplus_\alpha (A_\alpha \otimes B)$$

3. $$A \otimes B \cong B \otimes A \quad \textit{via the map } a \otimes b \mapsto b \otimes a$$

4. $$\mathbf{Z} \otimes A \cong A \quad \textit{via the map } 1 \otimes a \mapsto a$$

5. *For sets S and T*

   $$\mathbf{Z}[S] \otimes \mathbf{Z}[T] \cong \mathbf{Z}[S \times T] \quad \textit{via the map } s \otimes t \mapsto (s, t)$$

Proposition 137

If $A' \subset A$ and $B' \subset B$ are abelian groups then

$$(A/A') \otimes (B/B') \cong \frac{A \otimes B}{i_{A'} \otimes \mathbf{1}_B(A' \otimes B) + \mathbf{1}_A \otimes i_{B'}(A \otimes B')}$$

Proof.

- Let

$$q_1 : A \to A/A'$$
$$q_2 : B \to B/B'$$

  be the quotient maps
- Then by Lemma 135 we have

$$q_1 \otimes q_2 : A \otimes B \to (A/A') \otimes (B/B')$$

- Claim 1: $A' \otimes B \subset \ker q_1 \otimes q_2$. If $a' \in A'$ and $b \in B$ then

$$\begin{aligned} q_1 \otimes q_2(a' \otimes b) &= q_1(a') \otimes q_2(b) \\ &= 0 \otimes q_2(b) = 0 \end{aligned}$$

Proof of Proposition 137 *(continued)*.

$$q_1 : A \to A/A'$$

$$q_2 : B \to B/B'$$

- Similarly $A \otimes B' \subset \ker q_1 \otimes q_2$.
- Let $K = i_{A'} \otimes \mathbf{1}_B(A' \otimes B) + \mathbf{1}_A \otimes i_{B'}(A \otimes B')$
- $q_1 \otimes q_2(K) = 0$ so $q_1 \otimes q_2$ induces homomorphism

$$Q : (A \otimes B)/K \to (A/A') \otimes (B/B')$$

- Define

$$r : (A/A') \times (B/B') \to (A \otimes B)/K$$

  by setting

$$r([a], [b]) = [a \otimes b]$$

- r is well-defined and bilinear.

Proof of Proposition 137 *(continued)*.

$$q_1 : A \to A/A'$$

$$q_2 : B \to B/B'$$

- By universal propoerty get homomorphism

$$R : (A/A') \otimes (B/B') \to (A \otimes B)/K$$

with

$$R([a] \otimes [b]) = [a \otimes b]$$

- $QR = \mathbf{1}$ and $RQ = \mathbf{1}$
- Hence

$$(A/A') \otimes (B/B') \cong \frac{A \otimes B}{i_{A'} \otimes \mathbf{1}_B(A' \otimes B) + \mathbf{1}_A \otimes i_{B'}(A \otimes B')}$$

□

Example 138

$$\begin{aligned}(\mathbf{Z}/n\mathbf{Z}) \otimes (\mathbf{Z}/m\mathbf{Z}) &\cong \frac{\mathbf{Z} \otimes \mathbf{Z}}{n\mathbf{Z} \otimes \mathbf{Z} + \mathbf{Z} \otimes m\mathbf{Z}} \\ &\cong \frac{\mathbf{Z}}{n\mathbf{Z} + m\mathbf{Z}} \\ &\cong \mathbf{Z}/\gcd(n, m)\mathbf{Z}\end{aligned}$$

For example

$$(\mathbf{Z}/6\mathbf{Z}) \otimes (\mathbf{Z}/6\mathbf{Z}) \cong \mathbf{Z}/6\mathbf{Z}$$

$$(\mathbf{Z}/15\mathbf{Z}) \otimes (\mathbf{Z}/12\mathbf{Z}) \cong \mathbf{Z}/3\mathbf{Z}$$

$$(\mathbf{Z}/9\mathbf{Z}) \otimes (\mathbf{Z}/8\mathbf{Z}) \cong \mathbf{Z}/\mathbf{Z} = 0$$

Right exactness of tensor product

Proposition 139 (Right exactness of tensor product)

If

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is exact then for all D

$$A \otimes D \xrightarrow{\alpha \otimes \mathbf{1}_D} B \otimes D \xrightarrow{\beta \otimes \mathbf{1}_D} C \otimes D \to 0$$

is exact.

Proof of Proposition 139.

- Define

$$\varphi : C \times D \to \frac{B \otimes D}{\alpha \otimes \mathbf{1}_D(A \otimes D)}$$

  by setting $\varphi(\beta(b), d) = [b \otimes d]$
- Claim: φ is well-defined

Proof of Proposition 139 *(continued)*.

- Suppose $\beta(b) = \beta(b')$
- Then $b - b' \in \ker \beta$ thus there is $a \in A$ with $\alpha(a) = b - b'$
- Thus

$$\begin{aligned} b \otimes d - b' \otimes d &= (b - b') \otimes d \\ &= \alpha(a) \otimes d \\ &= \alpha \otimes \mathbf{1}_D(a \otimes d) \end{aligned}$$

- so φ is well-defined.
- Claim 2: φ is bilinear

$$\begin{aligned} \varphi(\beta(b) + \beta(b'), d) &= \varphi(\beta(b + b'), d) \\ &= [(b + b') \otimes d] \\ &= [b \otimes d + b' \otimes d] \\ &= [b \otimes d] + [b' \otimes d] \\ &= \varphi(\beta(b), d) + \varphi(\beta(b'), d) \end{aligned}$$

Proof of Proposition 139 *(continued)*.

- $$\begin{aligned}\varphi(\beta(b), d + d') &= [b \otimes (d + d')] \\ &= [b \otimes d + b \otimes d'] \\ &= \varphi(\beta(b), d) + \varphi(\beta(b), d')\end{aligned}$$
- Universal property of tensor product gives
$$\Phi : C \otimes D \to \frac{B \otimes D}{\alpha \otimes \mathbf{1}_D(A \otimes D)}$$
- Claim 3: $\beta \otimes \mathbf{1}_D(\alpha \otimes \mathbf{1}_D(A \otimes D)) = 0$
$$\begin{aligned}\beta \otimes \mathbf{1}_D(\alpha \otimes \mathbf{1}_D(a \otimes d)) &= (\beta \circ \alpha) \otimes (\mathbf{1}_D \circ \mathbf{1}_D)(a \otimes d) \\ &= \beta\alpha(a) \otimes d \\ &= 0 \otimes d = 0\end{aligned}$$

Proof of Proposition 139 *(continued)*.

- So
$$\beta \otimes \mathbf{1}_D : B \otimes D \to C \otimes D$$
induces
$$\Theta : \frac{B \otimes D}{\alpha \otimes \mathbf{1}_D(A \otimes D)} \to C \otimes D$$
- $\Theta\Phi(\beta(b) \otimes d) = \Theta([b \otimes d]) = \beta \otimes d$
- $\Phi\Theta([b \otimes d]) = \Phi(\beta(b) \otimes d) = [b \otimes d]$
- So Φ is an isomorphism establishing exactness at $B \otimes D$.
- Surjectivity of β gives surjectivity of $\beta \otimes \mathbf{1}_D$ establishing exactness at $C \otimes D$.

$\square$

Example 140 (Tensor product is not left exact)

- Consider the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\times 5} \mathbf{Z} \to \mathbf{Z}/5\mathbf{Z}$$

- Tensor with $\mathbf{Z}/5\mathbf{Z}$ to get

$$0 \to \mathbf{Z} \otimes (\mathbf{Z}/5\mathbf{Z}) \xrightarrow{(\times 5)\otimes \mathbf{1}} \mathbf{Z} \otimes (\mathbf{Z}/5\mathbf{Z}) \to \mathbf{Z}/5\mathbf{Z} \otimes \mathbf{Z}/5\mathbf{Z}$$

- which becomes

$$0 \to \mathbf{Z}/5\mathbf{Z} \xrightarrow{0} \mathbf{Z}/5\mathbf{Z} \xrightarrow{\cong} \mathbf{Z}/5\mathbf{Z}$$

- No longer exact at leftmost $\mathbf{Z}/5\mathbf{Z}$

Tensor products and chain complexes

Given a chain complex (C, ∂)

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \cdots$$

and an abelian group G we get a new chain complex $(C \otimes G, \partial \otimes \mathbf{1}_G)$.

$$\cdots \xrightarrow{\partial_{n+1} \otimes \mathbf{1}_G} C_n \otimes G \xrightarrow{\partial_n \otimes \mathbf{1}_G} C_{n-1} \otimes G \xrightarrow{\partial_{n-1} \otimes \mathbf{1}_G} C_{n-2} \otimes G \xrightarrow{\partial_{n-2} \otimes \mathbf{1}_G} \cdots$$

Lemma 141

If C is a chain complex then $C \otimes G$ is a chain complex.

Proof.

$$(\partial_n \otimes \mathbf{1}_G) \circ (\partial_{n-1} \otimes \mathbf{1}_G) = (\partial_n \circ \partial_{n-1}) \otimes \mathbf{1}_G = 0 \otimes \mathbf{1}_G = 0$$

$\square$

Lemma 142

If

$$f : C \to D$$

is a chain map then

$$f \otimes \mathbf{1}_G : C \otimes G \to D \otimes G$$

is a chain map.

Proof.

$f : C \to D$ is a chain map so $\partial f = f\partial$

$$\begin{aligned}(f \otimes \mathbf{1}_G) \circ (\partial \otimes \mathbf{1}_G) &= (f \circ \partial) \otimes \mathbf{1}_G \\ &= (\partial \circ f) \otimes \mathbf{1}_G \\ &= (\partial \otimes \mathbf{1}_G) \circ (f \otimes \mathbf{1}_G)\end{aligned}$$

□

Lemma 143

If

$$T : C \to D$$

is a chain homotopy between chain maps $f, g : C \to D$ *then*

$$T \otimes \mathbf{1}_G : C \otimes G \to D \otimes G$$

is a chain homotopy between chain maps $f \otimes \mathbf{1}_G$ *and* $g \otimes \mathbf{1}_G$

Proof.

$T : C \to D$ is a chain homotopy so $\partial T + T\partial = g - f$

$$\begin{aligned}(\partial \otimes \mathbf{1}) \circ (T \otimes \mathbf{1}) + (T \otimes \mathbf{1}) \circ (\partial \otimes \mathbf{1}) &= (\partial T) \otimes \mathbf{1} + (T\partial) \otimes \mathbf{1} \\ &= (\partial T + T\partial) \otimes \mathbf{1} \\ &= (g - f) \otimes \mathbf{1} \\ &= g \otimes \mathbf{1} - f \otimes \mathbf{1}\end{aligned}$$

□

Homology with coefficients in G

Definition 144 (Homology with coefficients in G)

Given a chain complex C let

$$H_n(C; G) = H_n(C \otimes G)$$

If X is a space let

$$H_n(X; G) = H_n(C(X) \otimes G)$$

$$\tilde{H}_n(X; G) = H_n(\tilde{C}(X) \otimes G)$$

$$H_n^{\mathrm{CW}}(X; G) = H_n(C^{\mathrm{CW}}(X) \otimes G)$$

For (X, A) a topological pair let

$$H_n(X, A; G) = H_n(C(X, A) \otimes G)$$

Example 145 (Homology of $\mathbf{R}\mathrm{P}^4$)

Standard CW chain complex for $\mathbf{R}\mathrm{P}^4$ is

$$\cdots \to 0 \to \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{\times 0} \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{\times 0} \mathbf{Z} \xrightarrow{\times 0} 0$$

Tensoring with $\mathbf{Z}_2$ we get $C^{\mathrm{CW}}(\mathbf{R}\mathrm{P}^4) \otimes \mathbf{Z}_2$

$$\cdots \to 0 \to \mathbf{Z}_2 \xrightarrow{\times 0} \mathbf{Z}_2 \xrightarrow{\times 0} \mathbf{Z}_2 \xrightarrow{\times 0} \mathbf{Z}_2 \xrightarrow{\times 0} \mathbf{Z}_2 \xrightarrow{\times 0} 0$$

So

$$H_n^{\mathrm{CW}}(\mathbf{R}\mathrm{P}^4; \mathbf{Z}_2) \cong \begin{cases} \mathbf{Z}_2, & 0 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

Tensoring with $\mathbf{Q}$ we get $C^{\mathrm{CW}}(\mathbf{R}\mathrm{P}^4) \otimes \mathbf{Q}$

$$\cdots \to 0 \to \mathbf{Q} \underset{\cong}{\xrightarrow{\times 2}} \mathbf{Q} \xrightarrow{\times 0} \mathbf{Q} \underset{\cong}{\xrightarrow{\times 2}} \mathbf{Q} \xrightarrow{\times 0} \mathbf{Q} \xrightarrow{\times 0} 0$$

So

$$H_n^{\mathrm{CW}}(\mathbf{R}\mathrm{P}^4; \mathbf{Q}) \cong \begin{cases} \mathbf{Q}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Note that $C \otimes \mathbf{Z} = C$ so

$$H_n(C; \mathbf{Z}) \cong H_n(C)$$

Why consider homology with other coefficent groups?

1. If G is a right R-module then $H_n(C; G)$ is a right R-module
2. In particular if G is a field then $H_n(C; G)$ is a vector space over G.
3. Computation of $H_n(C; \mathbf{Z}_2)$ are often easier.

What advantages does the coefficient group $\mathbf{Z}$ have?

1. Homology groups with coefficients in $\mathbf{Z}$ are the strongest topological invariants

Homology with coefficients in $\mathbf{Z}$ are "universal" since they determine homology with coefficients in G for all abelian groups G.

Theorem 146 (Universal coefficient theorem)

For C a chain complex there is a split short exact sequence

$$0 \to H_n(C) \otimes G \to H_n(C; G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0.$$

This sequence is natural in that if $C \to D$ is a chain map then we get commutative

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H_n(C) \otimes G \arrow[r] \arrow[d] & H_n(C; G) \arrow[r] \arrow[d] & \mathrm{Tor}(H_{n-1}(C), G) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & H_n(D) \otimes G \arrow[r] & H_n(D; G) \arrow[r] & \mathrm{Tor}(H_{n-1}(D), G) \arrow[r] & 0
\end{tikzcd}$$

The Tor functor

Definition 147 (Free resolution)

A **free resolution** of the abelian group A is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to A \to 0$$

where each F_i is free abelian.

Example 148 (Two free resolutions of $\mathbf{Z}_5$)

$$\cdots \to 0 \to 0 \to \mathbf{Z} \xrightarrow{\times 5} \mathbf{Z} \to \mathbf{Z}_5 \to 0$$

$$\cdots \xrightarrow{\times 1} \mathbf{Z} \xrightarrow{\times 0} \mathbf{Z} \xrightarrow{\times 1} \mathbf{Z} \xrightarrow{\times 0} \mathbf{Z} \xrightarrow{\times 5} \mathbf{Z} \to \mathbf{Z}_5 \to 0$$

Theorem 149 (Existence of free resolutions)

Every abelian group A has a free resolution

$$\cdots \to 0 \to R \to F \to A \to 0$$

where F and R are free abelian.

Theorem 150

Every subgroup of a free abelian group is a free abelian group.

Proof of Theorem 149.

We have a surjection

$$\mathbf{Z}[A] \xrightarrow{p} A$$

given by $p(a) = a$. Let $F = \mathbf{Z}[A]$ and $R = \ker p$.

Then

$$\cdots \to 0 \to R \to F \xrightarrow{p} A \to 0$$

is a free resolution of A. $\square$

Definition 151 (The Tor functor)

Let A and B be abelian groups and

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to A \to 0$$

be a free resolution of A.

Let $\partial_0 : F_0 \to 0$ be the 0 map so we have a chain complex (F, ∂)

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0$$

Tensoring with B we get the chain complex $F \otimes B$

$$\cdots \xrightarrow{\partial_3 \otimes \mathbf{1}_B} F_2 \otimes B \xrightarrow{\partial_2 \otimes \mathbf{1}_B} F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B \xrightarrow{\partial_0 \otimes \mathbf{1}_B} 0$$

$$\mathrm{Tor}_n(A, B) = H_n(F \otimes B)$$

Theorem 152 (Tor is well-defined)

$\mathrm{Tor}_n(A, B)$ *is independent of the free resolution* F *of* A.

Lemma 153 (Free abelian groups are projective)

Suppose we have commutative

$$\begin{tikzcd} & F \arrow[dl, dotted, "\exists\psi"'] \arrow[d, "\varphi"] \arrow[dr, "0"] & \\ M' \arrow[r, "i"'] & M \arrow[r, "j"'] & M'' \end{tikzcd}$$

1. *If F is a free abelian group*
2. *$M' \xrightarrow{i} M \xrightarrow{j} M''$ is exact*

There is a homomorphism ψ making the diagram commute.

Proof.

- Let $\{e_\alpha\}$ be a free basis for F
- $j\varphi(e_\alpha) = 0(e_\alpha) = 0$ so $\varphi(e_\alpha) \in \ker j = \operatorname{Im} i$
- Thus there is $m'_\alpha \in M'$ such that $i(m'_\alpha) = \varphi(e_\alpha)$
- Let $\psi(e_\alpha) = m'_\alpha$

$\square$

We will prove Theorem 152 using the following lemma

Lemma 154

If F and F' are two free resolutions of A then there is a chain map $f : F \to F'$ which is a chain homotopy equivalence.

Proof.

- We have

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_2"] & F_1 \arrow[r, "\partial_1"] & F_0 \arrow[r, "\varepsilon"] & A \arrow[r, "0"] \arrow[d, "\mathbf{1}_A"] & 0 \\
\cdots \arrow[r, "\partial_2'"] & F_1' \arrow[r, "\partial_1'"] & F_0' \arrow[r, "\varepsilon'"] & A \arrow[r, "0"] & 0
\end{tikzcd}$$

- Apply Lemma 153 (free ab. grps. are proj.) with $\varphi = \mathbf{1}_A \varepsilon$
- Get $\psi : F_0 \to F_0'$ with $\varepsilon' \psi = \mathbf{1}_A \varepsilon$
- Let $f_0 = \psi$
- Now apply Lemma 153 with $\varphi = f_0 \partial_1$ to get $f_1 : F_1 \to F_1'$
- Continue inductively to get chain map $f : F \to F'$

Proof of Lemma 154 *(continued)*.

- Now we will show that the chain homotopy type of f is unique.
- That is if $g : F \to F'$ is another chain map extending $\mathbf{1}_A : A \to A$ then there is a chain homotopy T between f and g
- Let $\tau = g - f$
- Assume inductively that $\partial'_{n+1} T_n + T_{n-1}\partial_n = \tau_n$
- We have (non-commutating diagram)

$$\begin{tikzcd}
 & F_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\tau_{n+1}"'] & F_n \arrow[r, "\partial_n"] \arrow[d, "\tau_n"'] \arrow[dl, "T_n"'] & F_{n-1} \arrow[dl, "T_{n-1}"'] \\
F'_{n+2} \arrow[r, "\partial'_{n+2}"] & F'_{n+1} \arrow[r, "\partial'_{n+1}"] & F'_n &
\end{tikzcd}$$

- Consider the map $(\tau_{n+1} - T_n\partial_{n+1}) : F_{n+1} \to F'_{n+1}$

$$\begin{aligned}
\partial'_{n+1}(\tau_{n+1} - T_n\partial_{n+1}) &= \partial'_{n+1}\tau_{n+1} - \partial'_{n+1}T_n\partial_{n+1} \\
&= \partial'_{n+1}\tau_{n+1} - (\tau_n - T_{n-1}\partial_n)\partial_{n+1} \\
&= \partial'_{n+1}\tau_{n+1} - \tau_n\partial_{n+1} + T_{n-1}\partial_n\partial_{n+1} \\
&= 0
\end{aligned}$$

Proof of Lemma 154 *(continued)*.

- Apply Lemma 153 (free ab. grps. are proj.) with $\varphi = \tau_{n+1} - T_n \partial_{n+1}$
- Let $T_{n+1} = \psi : F_{n+1} \to F'_{n+2}$
- Then $\partial'_{n+2} T_{n+1} = \varphi = \tau_{n+1} - T_n \partial_{n+1}$
- Hence
$$\partial'_{n+2} T_{n+1} + T_n \partial_{n+1} = \tau_{n+1} = g_{n+1} - f_{n+1}$$
- We may start the induction in degree -2 where all groups are 0.
- Thus the chain homotopy class of $f : F \to F'$ is unique.
- Similarly we get $f' : F' \to F$
- $\mathbf{1}_F : F \to F$ extends $\mathbf{1}_A : A \to A$
- so $f' \circ f$ is chain homotopic to $\mathbf{1}_F$
- That is, f is a chain homotopy equivalence.

$\square$

Now we finish proof that $\mathrm{Tor}_n(A, B)$ is independent of free resolution of A.

Proof of Theorem 152 (Tor is well-defined).

- Let F and F' be two free resolutions of A
- By Lemma 154 we have a chain homotopy equivalence
$$f : F \to F'$$
- Hence there is a chain map $f' : F' \to F$ such that $f' \circ f$ is chain homotopic to $\mathbf{1}_F$
- Tensoring with B we get $(f' \otimes \mathbf{1}_B) \circ (f \otimes \mathbf{1}_B)$ is chain homotopic to $\mathbf{1}_F \otimes B$
- Thus
$$(f \otimes \mathbf{1}_B)_* : H_n(F \otimes B) \to H_n(F' \otimes B)$$
is an isomorphism
- $\mathrm{Tor}_n(A, B) = H_n(F \otimes B) \cong H_n(F' \otimes B)$ is well-defined

$\square$

- As we saw in Lemma 149 every abelian group A has a free resolution
$$\cdots \to 0 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \to 0$$
- So we have the chain complexs F
$$\cdots \to 0 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0$$
- and $F \otimes B$
$$\cdots \to 0 \xrightarrow{\partial_2 \otimes \mathbf{1}_B} F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B \xrightarrow{\partial_0 \otimes \mathbf{1}_B} 0$$
- $$\mathrm{Tor}_n(A, B) = H_n(F \otimes B) = \begin{cases} \frac{F_0 \otimes B}{\mathrm{Im}(\partial_1 \otimes \mathbf{1}_B)}, & n = 0 \\ \ker(\partial_1 \otimes \mathbf{1}_B), & n = 1 \\ 0, & n \neq 0, 1 \end{cases}$$
- We can say more about $\mathrm{Tor}_0(A, B)$.

- We have the exact sequence

$$F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \to 0$$

- Which remains exact after tensoring with B so we get exact

$$F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes \mathbf{1}_B} A \otimes B \to 0$$

- Hence

$$\mathrm{Tor}_0(A, B) \cong \frac{F_0 \otimes B}{\mathrm{Im}(\partial_1 \otimes \mathbf{1}_B)} \cong A \otimes B$$

- Since $\mathrm{Tor}_1(A, B)$ is the only new object we define

Definition 155

$$\mathrm{Tor}(A, B) = \mathrm{Tor}_1(A, B)$$

- Note that if we have exact

$$0 \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \to 0$$

- Then we have exact

$$F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes \mathbf{1}_B} A \otimes B \to 0$$

- and hence exact

$$0 \to \ker(\partial_1 \otimes \mathbf{1}_B) \to F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes \mathbf{1}_B} A \otimes B \to 0$$

- and hence exact

$$0 \to \mathrm{Tor}(A, B) \to F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes \mathbf{1}_B} A \otimes B \to 0$$

- In particular if $\mathrm{Tor}(A, B) = 0$ then tensoring with B preserves exactness.

Example 156

Let's compute $\mathrm{Tor}(\mathbf{Z}_{60}, \mathbf{Z}_{42})$

- Free resolution F of $\mathbf{Z}_{60}$

$$\cdots \to 0 \to \mathbf{Z} \xrightarrow{\times 60} \mathbf{Z} \to \mathbf{Z}_{60} \to 0$$

- $F \otimes \mathbf{Z}_{42}$ is

$$0 \to \mathbf{Z} \otimes \mathbf{Z}_{42} \xrightarrow{(\times 60)\otimes \mathbf{1}} \mathbf{Z} \otimes \mathbf{Z}_{42} \to 0$$

- Simplifying

$$0 \to \mathbf{Z}_{42} \xrightarrow{\times 60} \mathbf{Z}_{42} \to 0$$

- Hence

$$\mathrm{Tor}(\mathbf{Z}_{60}, \mathbf{Z}_{42}) \cong \ker(\times 60) \cong \frac{\mathbf{Z}}{42\mathbf{Z} + 60\mathbf{Z}} \cong \frac{\mathbf{Z}}{\gcd(42, 60)\mathbf{Z}} \cong \mathbf{Z}_6$$

Proposition 157 (Properties of Tor)

1. $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$
2. $\mathrm{Tor}(\oplus_\alpha A_\alpha, B) \cong \oplus_\alpha \mathrm{Tor}(A_\alpha, B)$
3. $\mathrm{Tor}(A, B) = 0$ *if A or B is free or torsion free.*
4. $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(A_{\mathrm{tor}}, B)$ *where A_{tor} is the torsion subgroup of A.*
5. $\mathrm{Tor}(\mathbf{Z}_n, A) \cong \ker\left(A \xrightarrow{\times n} A\right)$
6. *The short exact sequence*

$$0 \to B \to C \to D \to 0$$

yeilds a natural exact sequence

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \\ \to A \otimes B \to A \otimes C \to A \otimes D \to 0$$

Proof of Proposition 157.

2. $\mathrm{Tor}(\oplus_\alpha A_\alpha, B) \cong \oplus_\alpha \mathrm{Tor}(A_\alpha, B)$
   - Let F_α be a free resolution of A_α
   - Then $\oplus_\alpha F_\alpha$ is a free resolution of A_α
   - $$\begin{aligned}\mathrm{Tor}(\oplus_\alpha A_\alpha, B) &\cong H_1((\oplus_\alpha F_\alpha) \otimes B) \\ &\cong H_1(\oplus_\alpha (F_\alpha \otimes B)) \\ &\cong \oplus_\alpha H_1(F_\alpha \otimes B) \\ &\cong \oplus_\alpha \mathrm{Tor}(A_\alpha, B)\end{aligned}$$

5. $\mathrm{Tor}(\mathbf{Z}_n, A) \cong \ker\left(A \xrightarrow{\times n} A\right)$
   - Use the free resolution of $\mathbf{Z}_n$
   $$\cdots \to 0 \to \mathbf{Z} \xrightarrow{\times n} \mathbf{Z} \to A \to 0$$
   - Tensoring with A simplifies to
   $$\cdots \to 0 \to A \xrightarrow{(\times n)} A \to 0$$

Proof of Proposition 157 *(continued)*.

3. $\mathrm{Tor}(A, B) = 0$ if A or B is free (we will address torsion free later)
   - Suppose A is free.
   - Use the free resolution of A

$$\cdots \to 0 \to 0 \to A \to A \to 0$$

   - $\mathrm{Tor}(A, B) = \ker(0 \to 0) = 0$
   - Suppose B is $\mathbf{Z}$
   - Then tensoring an exact free resolution $0 \to F_1 \to F_0 \to A \to 0$ with B remains exact
   - Suppose $B \cong \oplus_\alpha \mathbf{Z}$
   - Then tensoring an exact free resolution $0 \to F_1 \to F_0 \to A \to 0$ with B is a direct sum of exact sequences which is exact.

Proof of Proposition 157 *(continued)*.

6. The short exact sequence

$$0 \to B \to C \to D \to 0$$

yeilds a natural exact sequence

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \\ \to A \otimes B \to A \otimes C \to A \otimes D \to 0$$

- Choose a free resolution F of the form $0 \to F_1 \to F_0 \to A \to 0$
- All the terms of F are free so tensoring $0 \to B \to C \to D \to 0$ with F_n remains exact.
- Get short exact sequence of chain complexes

$$0 \to (F \otimes B) \to (F \otimes C) \to (F \otimes D) \to 0$$

- Apply Snake Lemma to get natural exact sequence above.

Proof of Proposition 157 *(continued)*.

1. $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$
   - Consider the six term exact sequence from part 6 coming from the short exact
   $$0 \to F_1 \to F_0 \to B \to 0$$
   - $$0 \to \mathrm{Tor}(A, F_1) \to \mathrm{Tor}(A, F_0) \to \mathrm{Tor}(A, B) \to A \otimes F_1 \to A \otimes F_0 \to A \otimes B \to 0$$
   - F_0 and F_1 are free so by part 3 $\mathrm{Tor}(A, F_1) \cong \mathrm{Tor}(A, F_0) \cong 0$
   - $$0 \to \mathrm{Tor}(A, B) \to A \otimes F_1 \to A \otimes F_0 \to A \otimes B \to 0$$
   - $$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}(A, B) \arrow[r] & A \otimes F_1 \arrow[r] \arrow[d, "\cong"] & A \otimes F_0 \arrow[r] \arrow[d, "\cong"] & A \otimes B \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathrm{Tor}(B, A) \arrow[r] & F_1 \otimes A \arrow[r] & F_0 \otimes A \arrow[r] & B \otimes A \arrow[r] & 0 \end{tikzcd}$$

Proof of Proposition 157 *(continued)*.

1. $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$ *(continued)*

- $$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}(A,B) \arrow[r, "\alpha"] \arrow[d, dotted, "\gamma"] & A \otimes F_1 \arrow[r, "\beta"] \arrow[d, "\tau"', "\cong"] & A \otimes F_0 \arrow[r] \arrow[d, "\mu"', "\cong"] & A \otimes B \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathrm{Tor}(B,A) \arrow[r, "\alpha'"] & F_1 \otimes A \arrow[r, "\beta'"] & F_0 \otimes A \arrow[r] & B \otimes A \arrow[r] & 0 \end{tikzcd}$$
- We will define a homorphism $\gamma : \mathrm{Tor}(A,B) \to \mathrm{Tor}(B,A)$ preserving commutativity
- Let $x \in \mathrm{Tor}(A,B)$
- Claim: $\tau\alpha(x) \in \mathrm{Im}\,\alpha'$
- By commutativity $\beta'\tau\alpha(x) = \mu\beta\alpha(x) = \mu(0) = 0$
- So $\tau\alpha(x) \in \ker\beta' = \mathrm{Im}\,\alpha'$
- By injectivity of α' there is a unique $x' \in \mathrm{Tor}(B,A)$ with $\alpha'(x') = \tau\alpha(x)$
- Set $\gamma(x) = x'$.
- γ takes 0 to 0 and sums to sums so it is a homomorphism.
- $$\begin{tikzcd} 0 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] \arrow[d, "\cong"] & \mathrm{Tor}(A,B) \arrow[r] \arrow[d, "\gamma"] & A \otimes F_1 \arrow[r] \arrow[d, "\tau"', "\cong"] & A \otimes F_0 \arrow[d, "\mu"', "\cong"] \\ 0 \arrow[r] & 0 \arrow[r] & \mathrm{Tor}(B,A) \arrow[r] & F_1 \otimes A \arrow[r] & F_0 \otimes A \end{tikzcd}$$

Proof of Proposition 157 *(continued)*.

1. $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$ *(continued)*
   - Add some trivial groups and homomorphisms to get

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] \arrow[d, "\cong"] & \mathrm{Tor}(A,B) \arrow[r] \arrow[d, "\gamma"] & A \otimes F_1 \arrow[r] \arrow[d, "\tau"', "\cong"] & A \otimes F_0 \arrow[d, "\mu"', "\cong"] \\
0 \arrow[r] & 0 \arrow[r] & \mathrm{Tor}(B,A) \arrow[r] & F_1 \otimes A \arrow[r] & F_0 \otimes A
\end{tikzcd}$$

   - Now apply Five Lemma to show $\gamma : \mathrm{Tor}(A, B) \to \mathrm{Tor}(B, A)$ is an isomorphism.

Proof of Proposition 157 *(continued)*.

3. $\mathrm{Tor}(A, B) \cong 0$ if A or B is torsion free.
   - Applying part 1 assume B is torsion free.
   - Let
   $$0 \to F_1 \xrightarrow{\partial_1} F_0 \to A$$
   be a free resolution of A.
   - Get exact
   $$0 \to \mathrm{Tor}(A, B) \to F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathbf{1}_B} F_0 \otimes B$$
   - Claim $\partial_1 \otimes \mathbf{1}_B$ is injective.
   - Suppose $\sum_i f_i \otimes b_i \in \ker \partial_1 \otimes \mathbf{1}_B$
   - Then then $\sum_i (\partial_1 f_i) \otimes b_i = 0$ in $F_0 \otimes B$.
   - Hence in $\mathbf{Z}[F_0 \times B]$
   $$\sum_i (\partial_1 f_i, b_i) = \sum_j (f_j^0, b_j^1 + b_j^2) - (f_j^0, b_j^1) - (f_j^0, b_j^2)$$
   $$+ \sum_k (f_k^1 + f_k^2, b_k^0) - (f_k^1, b_k^0) - (f_k^2, b_k^2)$$
   - Let $B_0 \subset B$ be the subgroup generated by the finite set $\{b_i, b_k^0, b_j^1, b_j^2\}$

Proof of Proposition 157 *(continued)*.

3. $\mathrm{Tor}(A, B) \cong 0$ if A or B is torsion free. *(continued)*
   - Hence in $\mathbf{Z}[F_0 \times B_0]$

$$\sum_i (\partial_1 f_i, b_i) = \sum_j (f_j^0, b_j^1 + b_j^2) - (f_j^0, b_j^1) - (f_j^0, b_j^2) + \sum_k (f_k^1 + f_k^2, b_k^0) - (f_k^1, b_k^0) - (f_k^2, b_k^2)$$

   - Therefore in $F_0 \otimes B_0$

$$\sum_i (\partial_1 f_i) \otimes b_i = 0$$

   - Let $B_0 \subset B$ is torsion free and finitely generated so free abelian.
   - Hence

$$F_1 \otimes B_0 \xrightarrow{\partial_1 \otimes \mathbf{1}_{B_0}} F_0 \otimes B_0$$

     is injective.
   - Thus in $F_1 \otimes B_0$

$$\sum_i f_i \otimes b_i = 0$$

Proof of Proposition 157 *(continued)*.

3. $\mathrm{Tor}(A, B) \cong 0$ if A or B is torsion free. *(continued)*
   - Hence in $\mathbf{Z}[F_1 \times B_0]$

$$\sum_i (f_i, b_i) = \sum_n (f_n^3, b_n^4 + b_n^5) - (f_n^3, b_n^4) - (f_n^3, b_n^5)$$
$$+ \sum_k (f_m^4 + f_k^5, b_m^3) - (f_m^4, b_m^3) - (f_m^4, b_m^3)$$

   - This equality holds in $\mathbf{Z}[F_1 \times B]$
   - Thus in $F_1 \otimes B$

$$\sum_i f_i \otimes b_i = 0$$

   - It follows that $\mathrm{Tor}(A, B) = \ker(\partial_1 \otimes \mathbf{1}_B) = 0$

Proof of Proposition 157 *(continued)*.

4. $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(A_{\mathrm{tor}}, B)$ where A_{tor} is the torsion subgroup of A.
   - We have the exact sequence $0 \to A_{\mathrm{tor}} \to A \to (A/A_{\mathrm{tor}}) \to 0$
   - Apply part 6 to get exact

$$0 \to \mathrm{Tor}(B, A_{\mathrm{tor}}) \to \mathrm{Tor}(B, A) \to \mathrm{Tor}(B, A/A_{\mathrm{tor}}) \\ \to B \otimes A_{\mathrm{tor}} \to B \otimes A \to B \otimes A/A_{\mathrm{tor}} \to 0$$

   - A/A_{tor} is torsion free so by part 3 $\mathrm{Tor}(B, A/A_{\mathrm{tor}}) = 0$
   - Get exact

$$0 \to \mathrm{Tor}(B, A_{\mathrm{tor}}) \to \mathrm{Tor}(B, A) \to 0$$

   - Thus $\mathrm{Tor}(B, A_{\mathrm{tor}}) \cong \mathrm{Tor}(B, A)$
   - and by part 1 $\mathrm{Tor}(A_{\mathrm{tor}}, B) \cong \mathrm{Tor}(A, B)$

$\square$

Universal coefficient theorem

Theorem 158 (Universal coefficient theorem)

If C a chain complex with each C_n free abelian there is a split short exact sequence

$$0 \to H_n(C) \otimes G \to H_n(C; G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0.$$

In particular

$$H_n(C; G) \cong \big(H_n(C) \otimes G\big) \oplus \mathrm{Tor}(H_{n-1}(C), G)$$

The short exact sequence is natural in that if $C \to D$ is a chain map then we get commutative

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H_n(C) \otimes G \arrow[r] \arrow[d] & H_n(C; G) \arrow[r] \arrow[d] & \mathrm{Tor}(H_{n-1}(C), G) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & H_n(D) \otimes G \arrow[r] & H_n(D; G) \arrow[r] & \mathrm{Tor}(H_{n-1}(D), G) \arrow[r] & 0
\end{tikzcd}$$

Proof of Theorem 158.

- Let C be a chain complex with each C_n free abelain
- Let $Z_n = \ker \partial_n \subset C_n$ be the n-cycles
- Let $B_n = \operatorname{Im} \partial_{n+1} \subset C_n$ be the n-boundaries
- View Z and B as chain complexes.
- Z_n and B_n are subgroups of free abelian groups and hence free abelian.
- Thus we have exact

$$0 \to Z_n \to C_n \to B_{n-1} \to 0$$

- View above as a free resolution of B_{n-1}
- Get exact

$$0 \to \operatorname{Tor}(B_{n-1}, G) \to Z_n \otimes G \to C_n \otimes G \to B_{n-1} \otimes G \to 0$$

- B_{n-1} is free so $\operatorname{Tor}(B_{n-1}, G) = 0$

Proof of Theorem 158 *(continued)*.

- B_{n-1} is free so $\mathrm{Tor}(B_{n-1}, G) = 0$
- Thus we have exact

$$0 \to Z_n \otimes G \to C_n \otimes G \to B_{n-1} \otimes G \to 0$$

- And hence exact sequence of chain complexes

$$0 \to Z \otimes G \to C \otimes G \to B \otimes G \to 0$$

- Apply Snake Lemma and fact that $\partial = 0$ for Z and B to get exact

$$\cdots \to B_n \otimes G \xrightarrow{\kappa} Z_n \otimes G \to H_n(C \otimes G) \\ \to B_{n-1} \otimes G \xrightarrow{\kappa} Z_{n-1} \otimes G \to \cdots$$

where κ is the connecting homomorphism

Proof of Theorem 158 *(continued)*.

- What is κ?
- We have

$$\begin{tikzcd}
0 \arrow[r] & Z_n \otimes G \arrow[r] \arrow[d] & C_n \otimes G \arrow[r, "\partial_n \otimes \mathbf{1}"] \arrow[d, "\partial_n \otimes \mathbf{1}"] & B_{n-1} \otimes G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_{n-1} \otimes G \arrow[r] & C_{n-1} \otimes G \arrow[r, "\partial_{n-1} \otimes \mathbf{1}"] & B_{n-2} \otimes G \arrow[r] & 0
\end{tikzcd}$$

- Given $b_{n-1} \otimes g \in B_{n-1} \otimes G$
- There is $c_n \in C_n$ such that $\partial_n c_n = b_{n-1}$
- Maps to $b_{n-1} \otimes g$ now viewed as a element of $C_{n-1} \otimes G$
- Which is the image of the element $b_{n-1} \otimes g \in Z_{n-1} \otimes G$
- Thus the connecting homomorphism is $\kappa = i_{n-1} \otimes \mathbf{1}$ where

$$i_{n-1} : B_{n-1} \to Z_{n-1}$$

is inclusion

Proof of Theorem 158 *(continued)*.

- From long exact sequence

$$\cdots \to B_n \otimes G \xrightarrow{i_n \otimes \mathbf{1}} Z_n \otimes G \to H_n(C \otimes G) \to B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes \mathbf{1}} Z_{n-1} \otimes G \to \cdots$$

- Get short exact sequence

$$0 \to \frac{Z_n \otimes G}{\mathrm{Im}(i_n \otimes \mathbf{1})} \to H_n(C \otimes G) \to \ker(i_{n-1} \otimes \mathbf{1}) \to 0$$

- By the definition of $H_{n-1}(C)$ we have a free resolution

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0$$

- Tensoring with G we get exact

$$0 \to \mathrm{Tor}(H_n(C), G) \to B_n \otimes G \xrightarrow{i_n \otimes \mathbf{1}} Z_n \otimes G \to H_n(C) \otimes G \to 0$$

Proof of Theorem 158 *(continued)*.

- Apply exactness of

$$0 \to \mathrm{Tor}(H_n(C), G) \to B_n \otimes G \xrightarrow{i_n \otimes \mathbf{1}} Z_n \otimes G \to H_n(C) \otimes G \to 0$$

- To short exact sequence

$$0 \to \frac{Z_n \otimes G}{\mathrm{Im}(i_n \otimes \mathbf{1})} \to H_n(C \otimes G) \to \ker(i_{n-1} \otimes \mathbf{1}) \to 0$$

- We see that

$$\frac{Z_n \otimes G}{\mathrm{Im}(i_n \otimes \mathbf{1})} \cong H_n(C) \otimes G$$

- and

$$\ker(i_{n-1} \otimes \mathbf{1}) \cong \mathrm{Tor}(H_n(C), G)$$

- Conclude with exactness of

$$0 \to H_n(C) \otimes G \to H_n(C; G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0.$$

□

Exercise 159 (HW7)

- *Hatcher pg. 155-159 Problems 19, 24, 40*
- *Hatcher pg. 267 Problems 1, 2*