Math 757
Homology theory

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Proof of the Künneth Formula

Proof of the Künneth Formula (for spaces).

- Given spaces X and Y we wish to show that we have a natural exact sequence

$$0 \to \bigoplus_i H_i(X) \otimes H_{n-i}(Y) \to H_n(X \times Y)$$
$$\to \bigoplus_i \mathrm{Tor}(H_i(X), H_{n-i-1}(Y)) \to 0$$

- By Eilenberg-Zilber Theorem we have a chain homotopy

$$C(X \times Y) \simeq C(X) \otimes C(Y)$$

- Hence

$$H_n(X \times Y) = H_n(C(X \times Y)) \cong H_n(C(X) \otimes C(Y))$$

Proof of the Künneth Formula *(continued)*.

- By the Künneth Formula for chain complexes we have natural exact

$$0 \to \bigoplus_i H_i(C(X)) \otimes H_{n-i}(C(Y)) \to H_n(C(X) \otimes C(Y))$$
$$\to \bigoplus_i \mathrm{Tor}(H_i(C(X)), H_{n-i-1}(C(Y))) \to 0$$

- Applying Eilenberg-Zilber we get

$$0 \to \bigoplus_i H_i(C(X)) \otimes H_{n-i}(C(Y)) \to H_n(C(X \times Y))$$
$$\to \bigoplus_i \mathrm{Tor}(H_i(C(X)), H_{n-i-1}(C(Y))) \to 0$$

- Finally by definition of sing. simp. homology we conclude

$$0 \to \bigoplus_i H_i(X) \otimes H_{n-i}(Y) \to H_n(X \times Y)$$
$$\to \bigoplus_i \mathrm{Tor}(H_i(X), H_{n-i-1}(Y) \to 0$$

$\square$

We must prove the Eilenberg-Zilber Theorem which says that for spaces X and Y there is a chain homotopy

$$C(X \times Y) \simeq C(X) \otimes C(Y)$$

Notation for product spaces

For a product space $X \times Y$ we have continuous projection maps

$$p_0 : X \times Y \to X$$

and

$$p_1 : X \times Y \to Y$$

and given continuous $f : W \to X$ and $g : W \to Y$ let

$$f \times g : W \to X \times Y$$

be the continuous map

$$f \times g(w) = (f(w), g(w))$$

Hence for any $u : W \to X \times Y$ we have $u = (p_0 u) \times (p_1 u)$

Proof of Eilenberg-Zilber.

- First we will apply the Acyclic Model Theorem to get a chain map
$$\varphi : C(X \times Y) \to C(X) \otimes C(Y)$$
- Let $\mathcal{C}$ be the category **Top** $\times$ **Top** of ordered pairs of spaces with morphisms from (X, Y) to (Z, W) given by pairs of maps (f, g) with $f : X \to Z$ and $g : Y \to W$ continuous.
- Let T : **Top** $\times$ **Top** $\to$ **Chain** be the functor
$$T(X, Y) = C(X \times Y)$$
and given a morphism $(f, g) : (X, Y) \to (Z, W)$ and a singular simplex $\sigma : \Delta^n \to X \times Y$ define
$$T(f, g) : C(X \times Y) \to C(Z \times W)$$
$$T(f, g)(\sigma) = (fp_0\sigma) \times (gp_1\sigma)$$

Proof of Eilenberg-Zilber *(continued)*.

- Let $S : \mathbf{Top} \times \mathbf{Top} \to \mathbf{Chain}$ be the functor

$$S(X, Y) = C(X) \otimes C(Y)$$

and given a morphism $(f, g) : (X, Y) \to (Z, W)$ and a singular simplices $\sigma : \Delta^p \to X$ and $\eta : \Delta^q \to Y$ define

$$S(f, g) : C(X) \otimes C(Y) \to C(Z) \otimes C(W)$$

$$S(f, g)(\sigma \otimes \eta) = (f\sigma) \otimes (g\eta)$$

Proof of Eilenberg-Zilber *(continued)*.

- Set $M_n = (\Delta^n, \Delta^n)$
- Choose models for $\mathcal{C}$ the singleton sets

$$\mathcal{M}_n = \{M_n\}$$

- Choose $e_n \in T_n(M_n) = C_n(\Delta^n \times \Delta^n)$ to be the n-chain

$$\mathbf{1} \times \mathbf{1} : \Delta^n \to \Delta^n \times \Delta^n$$

This is the **diagonal map** for Δ^n. Note that

$$p_0 e_n = p_1 e_n = \mathbf{1}_{\Delta^n}$$

Proof of Eilenberg-Zilber *(continued).*

- Claim that the functor T_n is free with models $\mathcal{M}_n$
  - In other words, we must show that $T(X, Y) = C(X \times Y)$ has free basis

$$\mathcal{B} = \left\{ T_n(f, g)(e_n) \middle| (f, g) : (\Delta^n, \Delta^n) \to (X, Y) \right\}$$

  - Given a singular simplex $\sigma : \Delta^n \to X \times Y$

$$\begin{aligned} T_n(p_0\sigma, p_1\sigma)(e_n) &= (p_0\sigma p_0 e_n) \times (p_1\sigma p_1 e_n) \\ &= (p_0\sigma \mathbf{1}_{\Delta^n}) \times (p_1\sigma \mathbf{1}_{\Delta^n}) \\ &= (p_0\sigma) \times (p_1\sigma) \\ &= \sigma \end{aligned}$$

  - Hence $\mathcal{B}$ contains the singular n-simplices $\mathcal{S}_n(X \times Y)$
  - $T_n(f, g)(e_n) = (fp_0 e_n) \times (gp_1 e_n) = f \times g$ is a map from Δ^n to $X \times Y$ so $\mathcal{S}_n(X \times Y)$ contains $\mathcal{B}$
  - Finally if $(f, g) \neq (f', g')$ then

$$T_n(f, g)(e_n) = f \times g \neq f' \times g' = T_n(f', g')(e_n)$$

Proof of Eilenberg-Zilber *(continued).*

- Claim that each M_n is S-acyclic
  - In other words, we must show that for $k > 0$ we have $H_k(S(M_n)) = 0$
  - $$\begin{aligned} H_k(S(M_n)) &= H_k(S(\Delta^n, \Delta^n)) \\ &= H_k(C(\Delta^n) \otimes C(\Delta^n)) \\ &\cong \bigoplus_i H_i(C(\Delta^n)) \otimes H_{k-i}(C(\Delta^n)) \\ &\quad \oplus \bigoplus_i \mathrm{Tor}(H_i(C(\Delta^n)), H_{k-i-1}(C(\Delta^n))) \\ &\cong \bigoplus_i H_i(\Delta^n) \otimes H_{k-i}(\Delta^n) \\ &\quad \oplus \bigoplus_i \mathrm{Tor}(H_i(\Delta^n), H_{k-i-1}(\Delta^n)) \\ &= \begin{cases} 0, & k \neq 0 \\ \mathbf{Z}, & k = 0 \end{cases} \end{aligned}$$

Proof of Eilenberg-Zilber *(continued)*.

- Claim there is a natural isomorphism $\Phi : H_0(T) \to H_0(S)$
  - In other words for each (X, Y) we have

$$\Phi_{(X,Y)} : H_0(T(X, Y)) \to H_0(S(X, Y))$$

  - $$\begin{aligned} H_0(S(X, Y)) &= H_0(C(X) \otimes C(Y)) \\ &\cong \bigoplus_i H_i(X) \otimes H_{-i}(Y) \\ &\quad \oplus \bigoplus_i \mathrm{Tor}(H_i(X), H_{-i-1}(Y)) \\ &= H_0(X) \otimes H_0(Y) \end{aligned}$$

  - $$\begin{aligned} H_0(T(X, Y)) &= H_0(C(X \times Y)) \\ &= H_0(X \times Y) \end{aligned}$$

  - Path components of $X \times Y$ are product of path components of X with path components of Y

Proof of Eilenberg-Zilber *(continued)*.

- Now apply Acyclic Models Theorem to get chain homotopy

$$\varphi : T \to S$$

  inducing an isomorphism $H_0(T) \cong H_0(S)$.
- Now we need a futher chain homotopy

$$\gamma : S \to T$$

  inducing the reverse isomorphism $H_0(S) \cong H_0(T)$.
- Here we need larger set of models

$$\mathcal{M}_n^S = \left\{ (\Delta^p, \Delta^q) \middle| p + q = n \right\}$$

- Let $e_{p,q}^S \in S_n(\Delta^p, \Delta^q) = \bigoplus_i C_i(\Delta^p) \otimes C_{n-i}(\Delta^q)$ be

$$e_{p,q}^S = \mathbf{1}_{\Delta^p} \otimes \mathbf{1}_{\Delta^q}$$

- S_n is free with models $\mathcal{M}_n^S$

Proof of Eilenberg-Zilber *(continued).*

- Claim that each $(\Delta^p, \Delta^q) \in \mathcal{M}_n^S$ is T-acyclic
  - In other words, we must show that for $k > 0$ we have $H_k(T(\Delta^p, \Delta^q)) = 0$
  - $$\begin{aligned} H_k(T(\Delta^p, \Delta^q)) &= H_k(C(\Delta^p \times \Delta^q)) \\ &= H_k(\Delta^p \times \Delta^q) \\ &\cong H_k(\{*\}) \\ &= \begin{cases} 0, & k \neq 0 \\ \mathbf{Z}, & k = 0 \end{cases} \end{aligned}$$

$\square$

This concludes the proof of the Künneth Formula (for spaces).

More applications of Acyclic Model Theorem

- We could have used Acyclic Model Theorem to prove homotopy invariance of H as follows
  - Let T : **Top** $\to$ **Chain** be

$$T(X) = C(X)$$

  - Let S : **Top** $\to$ **Chain** be

$$S(X) = C(X \times I)$$

  - $\mathcal{M}_n = \{\Delta^n\}$ with $e_n = \mathbf{1}_{\Delta^n} \in C_n(\Delta^n)$
  - T_n is free with models $\mathcal{M}_n$ (observe definitions)
  - Δ^n is S-acyclic (cone each sing. simplex to fixed $p \in \Delta^n \times I$ to show $H_k(\Delta^n \times I) = 0$ for $k > 0$).
  - Thus there is a unique chain homotopy type of chain maps

$$\gamma : C(X) \to C(X \times I)$$

    inducing the natural isomorphism $H_0(X) \cong H_0(X \times I)$.

More applications of Acyclic Model Theorem

- Let $h_0, h_1 : X \to X \times I$ be the maps

$$h_0(x) = (x, 0) \qquad h_1(x) = (x, 1)$$

- h_0 and h_1 induce $h_{0\sharp}, h_{1\sharp} : C(X) \to C(X \times I)$ which in turn induce the natural isomorphism $H_0(X) \cong H_1(X \times I)$
- Hence $h_{0\sharp}$ and $h_{1\sharp}$ are chain homotopic.
- No suppose

$$f_0, f_1 : X \to Y$$

  are homotopic
- Have homotopy $F : X \times I \to Y$ from f_0 to f_1 with $f_0 = F \circ h_0$ and $f_1 = F \circ h_1$
- Hence

$$f_{0\sharp} = F_\sharp \circ h_{0\sharp} \simeq F_\sharp \circ h_{1\sharp} = f_{1\sharp}$$

Review of Homotopy groups

Definition 172 (Homotopy for pairs)

(X, A) and (Y, B) topological pairs

$$f, g : (X, A) \to (Y, B)$$

are homotopic (resp. homotopic rel $X' \subset X$) if there is

$$F : (X \times I, A \times I) \to (Y, B)$$

with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. (resp. further $F|_{X'} = \mathbf{1}_{X'}$)

Denote the class of $f : (X, A) \to (Y, B)$ by $[f]$. (resp. $[f]_{X'}$)

Let

$$[(X, A); (Y, B)] = \left\{ [f] \middle| f : (X, A) \to (Y, B) \right\}$$

and

$$[(X, A); (Y, B)]_{X'} = \left\{ [f]_{X'} \middle| f : (X, A) \to (Y, B) \right\}$$

$$
\begin{aligned}
I^0 &= \{0\} \qquad \partial I^0 = \varnothing \\
I^n &= [0,1]^n \subset \mathbf{R}^n \\
\partial I^n &= \left\{ (x_1, \cdots, x_n) \middle| \text{ there is } i \text{ s.t. } x_i = 0 \text{ or } x_i = 1 \right\} \\
I^{n-1} &= \left\{ (x_1, \cdots, x_n) \middle| x_n = 0 \right\} \\
J^{n-1} &= \partial I^n - \mathrm{int}(I^{n-1})
\end{aligned}
$$

Definition 173 (Homotopy groups)

(X, x_0) a pointed space and $n \geq 0$

$$\pi_n(X, x_0) = [(I^n, \partial I^n); (X, x_0)]$$

(X, A, x_0) pointed pair of spaces and $n \geq 1$

$$\pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{n-1}); (X, A, x_0)]$$

Definition 174 (Group structure of homotopy groups)

$[f],[g] \in \pi_n(X, x_0)$ with $n \geq 1$ or $[f],[g] \in \pi_n(X, A, x_0)$ with $n \geq 2$

$$(f \cdot g)(x_1, \cdots, x_n) = \begin{cases} f(2x_1, x_2, \cdots, x_n), & x_1 \leq \frac{1}{2} \\ g(2x_1 - 1, x_2, \cdots, x_n), & x_1 \geq \frac{1}{2} \end{cases}$$

Setting

$$[f] \cdot [g] = [f \cdot g]$$

gives π_n a group structure.

- $\pi_0(X, x_0)$ and $\pi_1(X, A, x_0)$ are sets with distinguished "identity" element $[c_{x_0}]$
- $\pi_1(X, x_0)$ and $\pi_2(X, A, x_0)$ are nonabelian groups (in general)
- $\pi_n(X, x_0)$ and $\pi_{n+1}(X, A, x_0)$ are abelian for $n \geq 2$

$$D^n = \left\{ x \in \mathbf{R}^n \middle| |x| \leq 1 \right\}$$
$$S^{n-1} = \partial D^n = \left\{ x \in \mathbf{R}^n \middle| |x| = 1 \right\}$$
$$p = (1, 0, \cdots, 0)$$

Definition 175 (Homotopy groups (alt. def.))

(X, x_0) a pointed space and $n \geq 0$

$$\pi_n(X, x_0) = [(S^n, p); (X, x_0)]$$

(X, A, x_0) pointed pair of spaces and $n \geq 1$

$$\pi_n(X, A, x_0) = [(D^n, S^{n-1}, p); (X, A, x_0)]$$

Theorem 176 (Long exact sequence of homotopy groups)

(X, A) a pair of spaces. There is a natural long exact sequence

$$\cdots \to \pi_{n+1}(X, A, x_0) \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \cdots$$
$$\cdots \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{j_*} \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0)$$

- For last three maps "ker" means the preimage of $[c_{x_0}]$.
- i_* and j_* are obvious inclusion maps
- Given $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$

$$\partial[f] = [f|_{I^{n-1}}]$$

Theorem 177 (Long exact homotopy sequence of triple)

(X, A, B) a triple of spaces. There is a natural long exact sequence

$$\cdots \to \pi_{n+1}(X, A, x_0) \xrightarrow{\partial} \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \cdots$$
$$\cdots \xrightarrow{j_*} \pi_2(X, A, x_0) \xrightarrow{\partial} \pi_1(A, B, x_0) \xrightarrow{i_*} \pi_1(X, B, x_0) \xrightarrow{j_*} \pi_1(X, A, x_0)$$

Are homotopy groups a homology theory?

- Exactness Axiom holds (by above)
- Homotopy Axiom holds
- Dimension Axiom fails (but not badly)
- Additivity Axiom fails
- Excision Axiom fails (big problem)

Definition 178 (n-connectivity)

A space X is n-**connected** if

$$\pi_k(X, x_0) = 0 \quad \text{for all } k \leq n$$

A topological pair (X, A) is n-**connected** if

$$\pi_k(X, A, x_0) = 0 \quad \text{for all } k \leq n$$

- A space is 0-connected iff it is path connected
- A space is 1-connected iff it is simply connected
- S^n is $(n-1)$-connected but not n-connected

- Let X be a space and

$$G = \pi_1(X, x_0)$$

  Then we have a G-action on $\pi_n(X, x_0)$
- For $n \geq 2$ this makes $\pi_n(X, x_0)$ a G-module

Definition 179 (G-module)

A right or left G-module M is a abelian group with a right or left $\mathbf{Z}[G]$-module structure.

Definition 180 (G-coinvaraiants)

Given a G-module M, the G-coinvariants of M are

$$M_G = \frac{M}{\left\langle gm - m \middle| g \in G, m \in M \right\rangle}$$

or equivalently

$$M_G = M \otimes_{\mathbf{Z}[G]} \mathbf{Z}$$

where $\mathbf{Z}$ is the trivial G-module.

The Hurewicz Homomorphism

- Basic idea of Hurewicz Homorphism
- Given $[\alpha] \in \pi_n(X)$
- want $h([\alpha]) \in H_n(X)$
- have map
$$\alpha : (S^n, p) \to (X, X_0)$$
- Choose a generator $z_n \in H_n(S^n, p)$
- define
$$h([\alpha]) = \alpha_* z_n$$
- To get naturality of h must choose z_n compatibly for all n
- Some work to show h is a homomorphism.

- Goal: define Hurewicz homorphism
- Must inductively choose generators of

$$H_n(I^n, \partial I^n)$$

for all n.

- Choose $z_1 \in H_1(I, \partial I)$ to be unique element such that

$$\partial z_1 = [1] - [0] \in H_0(\partial I = \{0, 1\})$$

- By Excision have an isomorphism

$$j : H_{n-1}(I^{n-1}, \partial I^{n-1}) \cong H_{n-1}(\partial I^n, J^{n-1})$$

- Given $z_{n-1} \in H_{n-1}(I^{n-1}, \partial I^{n-1})$ choose $z_n \in H_n(I^n, \partial I^n)$ so that

$$\partial z_n = j z_{n-1}$$

Definition 181 (The Hurewicz Homomorphism)

The Hurewicz map is the function

$$h : \pi_n(X, A, x_0) \to H_n(X, A)$$

where given

$$[\alpha] \in \pi_n(X, A, x_0)$$

we let

$$h([\alpha]) = \alpha_* z_n \in H_n(X, A)$$

Letting $A = x_0$ and assuming $n \geq 1$ we get

$$h : \pi_n(X, A, x_0) \to H_n(X, x_0) \cong \tilde{H}_n(X) \cong H_n(X)$$

- Claim: h is well-defined
- Given $\alpha, \beta : (I^n, \partial I^n) \to (X, A)$
- if $\alpha \simeq \beta$
- then by Homotopy Axiom $\alpha_* = \beta_*$
- So $\alpha_* z_n = \beta_* z_n$

Theorem 182 (The Hurewicz Homomorphism)

For (X, A) a topological pair and $n \geq 1$ the Hurewicz map

$$h : \pi_n(X, A, x_0) \to H(X, A)$$

defined above satisfies the following:

1. *h is a homomorphism.*
2. *h is natural in that we have commutative*

$$\begin{tikzcd}
\cdots \arrow[r, "i_*"] & \pi_2(X, x_0) \arrow[r, "j_*"] \arrow[d, "h"] & \pi_2(X, A, x_0) \arrow[r, "\partial"] \arrow[d, "h"] & \pi_1(A, x_0) \arrow[r, "i_*"] \arrow[d, "h"] & \pi_1(X, x_0) \arrow[d, "h"] \\
\cdots \arrow[r, "i_*"] & H_2(X) \arrow[r, "j_*"] & H_2(X, A) \arrow[r, "\partial"] & H_1(A) \arrow[r, "i_*"] & H_1(X)
\end{tikzcd}$$

3. *h is also natural in that for $f : (X, A, x_0) \to (Y, B, y_0)$ we have commutative*

$$\begin{tikzcd}
\pi_n(X, A, x_0) \arrow[r, "f_*"] \arrow[d, "h"] & \pi_n(Y, B, y_0) \arrow[d, "h"] \\
H_n(X, A) \arrow[r, "f_*"] & H_n(Y, B)
\end{tikzcd}$$

Proof of Theorem 182.

3. Claim: Given $f : (X, A, x_0) \to (Y, B, y_0)$ we have commutative

$$\begin{tikzcd} \pi_n(X, A, x_0) \arrow[r, "f_\star"] \arrow[d, "h"] & \pi_n(Y, B, y_0) \arrow[d, "h"] \\ H_n(X, A) \arrow[r, "f_*"] & H_n(Y, B) \end{tikzcd}$$

- Given $\alpha : (I^n, \partial I^n) \to (X, A)$

$$\begin{aligned} h(f_\star[\alpha]) &= h([f\alpha]) \\ &= (f\alpha)_* z_n \\ &= f_*(\alpha_* z_n) \\ &= f_*(h([\alpha])) \end{aligned}$$

- Note : We have established commutativity of many squares of LES

Proof of Theorem 182 *(continued)*.

2. Claim: We have commutative

$$\begin{tikzcd} \pi_n(X, A, x_0) \arrow[r, "\partial"] \arrow[d, "h"] & \pi_{n-1}(A, x_0) \arrow[d, "h"] \\ H_n(X, A) \arrow[r, "\partial"] & H_{n-1}(A) \end{tikzcd}$$

- Given $\alpha : (I^n, \partial I^n) \to (X, A)$
- Let $\alpha' : (I^{n-1}, \partial I^{n-1}) \to (A, x_0)$ satisfy $\alpha' = \alpha|_{I^{n-1}}$
- Then $\partial[\alpha] = [\alpha']$
- If $j : (I^{n-1}, \partial I^{n-1}) \to (\partial I^n, J^{n-1})$ is inclusion we have $\alpha' = \alpha|_{\partial I^n} \circ j$

$$\begin{aligned} h(\partial[\alpha]) &= h([\alpha']) \\ &= \alpha'_* z_{n-1} \\ &= (\alpha|_{\partial I^n})_* j_* z_{n-1} \\ &= (\alpha|_{\partial I^n})_* \partial z_n \\ &= \alpha_* \partial z_n \\ &= \partial \alpha_* z_n = \partial h([\alpha]) \end{aligned}$$

Proof of Theorem 182 *(continued)*.

1. Claim: $h : \pi_n(X, A, x_0) \to H_n(X, A)$ is a homomorphism
   - Let $L \subset I^n$ be $\left\{(x_1, \cdots, x_n) \middle| x_1 \leq \frac{1}{2}\right\}$
   - Let $U \subset I^n$ be $\left\{(x_1, \cdots, x_n) \middle| x_1 \geq \frac{1}{2}\right\}$
   - Let $\partial L = L - \text{int}\, L$ and $\partial U = U - \text{int}\, U$
   - Consider inclusions

$$i_L : (L, \partial L) \to (I^n, \partial L \cup \partial U)$$

$$i_U : (U, \partial U) \to (I^n, \partial L \cup \partial U)$$

   - Get isomorphism

$$i_{L*} \oplus i_{U*} : H_n(L, \partial L) \oplus H_n(U, \partial U) \to H_n(I^n, \partial L \cup \partial U)$$

   - Consider "squeeze" maps

$$s_L : (I^n, \partial I^n) \to (L, \partial L) \qquad s_U : (I^n, \partial I^n) \to (U, \partial U)$$

   defined by

$$s_L(x_1, \cdots, x_n) = \left(\tfrac{x_1}{2}, \cdots, x_n\right) \quad s_U(x_1, \cdots, x_n) = \left(\tfrac{x_1+1}{2}, \cdots, x_n\right)$$

Proof of Theorem 182 *(continued)*.

1. $h : \pi_n(X, A, x_0) \to H_n(X, A)$ is a homomorphism *(continued)*
   - We have inculsion
   $$i : (I^n, \partial I^n) \to (I^n, \partial L \cup \partial U)$$
   - For $z_n \in H_n(I^n, \partial I^n)$
   $$i_* z_n = i_{L*} s_{L*} z_n + i_{U*} s_{U*} z_n$$
   - Now given $\alpha, \beta : (I^n, \partial I^n) \to (X, A)$
   $$\begin{aligned} h([\alpha] \cdot [\beta]) &= h([\alpha \cdot \beta]) \\ &= (\alpha \cdot \beta)_* z_n \\ &= \big((\alpha \cdot \beta) \circ i\big)_* z_n \\ &= (\alpha \cdot \beta)_* i_* z_n \\ &= (\alpha \cdot \beta)_* (i_{L*} s_{L*} + i_{U*} s_{U*}) z_n \\ &= \alpha z_n + \beta z_n \\ &= h([\alpha]) + h([\beta]) \end{aligned}$$

$\square$

Theorem 183

Let $G = \pi_1(X, x_0)$ *then for* $n \geq 1$

$$h : \pi_n(X, x_0) \to H_n(X)$$

factors through the quotient map

$$q : \pi_n(X, x_0) \to \pi_n(X, x_0)_G$$

to $\pi_n(X, x_0)_G$ *the* G*-coinvariants of* $\pi_n(X, x_0)$

Theorem 184

(X, A) *a topological pair. Let* $G = \pi_1(A, x_0)$ *then for* $n \geq 2$

$$h : \pi_n(X, A, x_0) \to H_n(X)$$

factors through the quotient map

$$q : \pi_n(X, A, x_0) \to \pi_n(X, A, x_0)_G$$

to $\pi_n(X, A, x_0)_G$ *the* G*-coinvariants of* $\pi_n(X, A, x_0)$

Proof of Theorems 183 and 184.

- $[f] \in \pi_1(X, x_0)$
- $[\alpha] \in \pi_n(X, x_0)$
- Standard argument gives explicit homotopy between α and $f \cdot \alpha$

$\square$

Exercise 185 (HW9 - Problem 1)

Decide whether or not for all spaces X and all $n \geq 1$ the Hurewicz homomorphism

$$h : \pi_n(X, x_0) \to H_n(X)$$

is surjective.

Exercise 186 (HW9 - Problem 2)

Hatcher pg. 389-392: 10, 11, 19, 20

You may use any results from Section 4.2 for these problems including the Hurewicz Theorem and Whitehead's Theorem