Math 757
Homology theory

March 3, 2011
Proof of the K"unneth Formula

Given spaces $X$ and $Y$ we wish to show that we have a natural exact sequence

$$0 \rightarrow \bigoplus_i H_i(X) \otimes H_{n-i}(Y) \rightarrow H_n(X \times Y)$$

$$\rightarrow \bigoplus_i \text{Tor}(H_i(X), H_{n-i-1}(Y)) \rightarrow 0$$

By Eilenberg-Zilber Theorem we have a chain homotopy

$$C(X \times Y) \cong C(X) \otimes C(Y)$$

Hence

$$H_n(X \times Y) = H_n(C(X \times Y)) \cong H_n(C(X) \otimes C(Y))$$
Proof of the Künneth Formula (continued).

- By the Künneth Formula for chain complexes we have natural exact

\[
0 \rightarrow \bigoplus_i H_i(C(X)) \otimes H_{n-i}(C(Y)) \rightarrow H_n(C(X) \otimes C(Y))
\]

\[
\rightarrow \bigoplus_i \text{Tor}(H_i(C(X)), H_{n-i-1}(C(Y))) \rightarrow 0
\]

- Applying Eilenberg-Zilber we get

\[
0 \rightarrow \bigoplus_i H_i(C(X)) \otimes H_{n-i}(C(Y)) \rightarrow H_n(C(X \times Y))
\]

\[
\rightarrow \bigoplus_i \text{Tor}(H_i(C(X)), H_{n-i-1}(C(Y))) \rightarrow 0
\]

- Finally by definition of sing. simp. homology we conclude

\[
0 \rightarrow \bigoplus_i H_i(X) \otimes H_{n-i}(Y) \rightarrow H_n(X \times Y)
\]

\[
\rightarrow \bigoplus_i \text{Tor}(H_i(X), H_{n-i-1}(Y)) \rightarrow 0
\]
We must prove the Eilenberg-Zilber Theorem which says that for spaces $X$ and $Y$ there is a chain homotopy

$$C(X \times Y) \cong C(X) \otimes C(Y)$$

**Notation for product spaces**

For a product space $X \times Y$ we have continuous projection maps

$$p_0 : X \times Y \rightarrow X$$

and

$$p_1 : X \times Y \rightarrow Y$$

and given continuous $f : W \rightarrow X$ and $g : W \rightarrow Y$ let

$$f \times g : W \rightarrow X \times Y$$

be the continuous map

$$f \times g(w) = (f(w), g(w))$$

Hence for any $u : W \rightarrow X \times Y$ we have $u = (p_0 u) \times (p_1 u)$
Proof of Eilenberg-Zilber.

- First we will apply the Acyclic Model Theorem to get a chain map

\[ \varphi : C(X \times Y) \rightarrow C(X) \otimes C(Y) \]

- Let \( C \) be the category \( \textbf{Top} \times \textbf{Top} \) of ordered pairs of spaces with morphisms from \((X, Y)\) to \((Z, W)\) given by pairs of maps \((f, g)\) with \( f : X \rightarrow Z \) and \( g : Y \rightarrow W \) continuous.

- Let \( T : \textbf{Top} \times \textbf{Top} \rightarrow \textbf{Chain} \) be the functor

\[ T(X, Y) = C(X \times Y) \]

and given a morphism \((f, g) : (X, Y) \rightarrow (Z, W)\) and a singular simplex \( \sigma : \Delta^n \rightarrow X \times Y \) define

\[ T(f, g) : C(X \times Y) \rightarrow C(Z \times W) \]

\[ T(f, g)(\sigma) = (fp_0 \sigma) \times (gp_1 \sigma) \]
Proof of Eilenberg-Zilber (continued).

- Let \( S : \textbf{Top} \times \textbf{Top} \rightarrow \textbf{Chain} \) be the functor

\[
S(X, Y) = C(X) \otimes C(Y)
\]

and given a morphism \((f, g) : (X, Y) \rightarrow (Z, W)\) and a singular simplices \(\sigma : \Delta^p \rightarrow X\) and \(\eta : \Delta^q \rightarrow Y\) define

\[
S(f, g) : C(X) \otimes C(Y) \rightarrow C(Z) \otimes C(W)
\]

\[
S(f, g)(\sigma \otimes \eta) = (f \sigma) \otimes (g \eta)
\]
Proof of Eilenberg-Zilber (continued).

- Set $M_n = (\Delta^n, \Delta^n)$
- Choose models for $C$ the singleton sets
  $$\mathcal{M}_n = \{M_n\}$$
- Choose $e_n \in T_n(M_n) = C_n(\Delta^n \times \Delta^n)$ to be the $n$-chain
  $$1 \times 1 : \Delta^n \to \Delta^n \times \Delta^n$$

  This is the **diagonal map** for $\Delta^n$. Note that
  $$p_0 e_n = p_1 e_n = 1_{\Delta^n}$$
Proof of Eilenberg-Zilber (continued).

• Claim that the functor $T_n$ is free with models $M_n$
  • In other words, we must show that $T(X, Y) = C(X \times Y)$ has free basis

$$B = \left\{ T_n(f, g)(e_n) \mid (f, g) : (\Delta^n, \Delta^n) \to (X, Y) \right\}$$

• Given a singular simplex $\sigma : \Delta^n \to X \times Y$

$$T_n(p_0\sigma, p_1\sigma)(e_n) = (p_0\sigma p_0 e_n) \times (p_1\sigma p_1 e_n)$$
$$= (p_0\sigma 1_{\Delta^n}) \times (p_1\sigma 1_{\Delta^n})$$
$$= (p_0\sigma) \times (p_1\sigma)$$
$$= \sigma$$

• Hence $B$ contains the singular $n$-simplices $S_n(X \times Y)$
• $T_n(f, g)(e_n) = (fp_0 e_n) \times (gp_1 e_n) = f \times g$ is a map from $\Delta^n$ to $X \times Y$ so $S_n(X \times Y)$ contains $B$
• Finally if $(f, g) \neq (f', g')$ then

$$T_n(f, g)(e_n) = f \times g \neq f' \times g' = T_n(f', g')(e_n)$$
Proof of Eilenberg-Zilber (continued).

- Claim that each $M_n$ is $S$-acyclic
  - In other words, we must show that for $k > 0$ we have $H_k(S(M_n)) = 0$

$$H_k(S(M_n)) = H_k(S(\Delta^n, \Delta^n))$$
$$= H_k(C(\Delta^n) \otimes C(\Delta^n))$$
$$\cong \bigoplus_i H_i(C(\Delta^n)) \otimes H_{k-i}(C(\Delta^n))$$
$$\bigoplus \bigoplus \text{Tor}(H_i(C(\Delta^n)), H_{k-i-1}(C(\Delta^n)))$$
$$\cong \bigoplus_i H_i(\Delta^n) \otimes H_{k-i}(\Delta^n)$$
$$\bigoplus \bigoplus \text{Tor}(H_i(\Delta^n), H_{k-i-1}(\Delta^n))$$
$$= \begin{cases} 
0, & k \neq 0 \\
\mathbb{Z}, & k = 0 
\end{cases}$$
Proof of Eilenberg-Zilber (continued).

- Claim there is a natural isomorphism $\Phi : H_0(T) \to H_0(S)$
  - In other words for each $(X, Y)$ we have
    $$\Phi_{(X,Y)} : H_0(T(X, Y)) \to H_0(S(X, Y))$$

  $H_0(S(X, Y)) = H_0(C(X) \otimes C(Y))$
  \[\cong \bigoplus_i H_i(X) \otimes H_{-i}(Y)\]
  \[\oplus \bigoplus_i \text{Tor}(H_i(X), H_{-i-1}(Y))\]
  \[= H_0(X) \otimes H_0(Y)\]

- $H_0(T(X, Y)) = H_0(C(X \times Y))$
  \[= H_0(X \times Y)\]

- Path components of $X \times Y$ are product of path components of $X$ with path components of $Y$
Proof of Eilenberg-Zilber \((continued)\).

- Now apply Acyclic Models Theorem to get chain homotopy
  \[ \varphi : T \to S \]
  inducing an isomorphism \( H_0(T) \cong H_0(S) \).
- Now we need a further chain homotopy
  \[ \gamma : S \to T \]
  inducing the reverse isomorphism \( H_0(S) \cong H_0(T) \).
- Here we need larger set of models
  \[ \mathcal{M}_n^S = \left\{ (\Delta^p, \Delta^q) \middle| p + q = n \right\} \]
- Let \( e_{p,q}^S \in S_n(\Delta^p, \Delta^q) = \bigoplus_i C_i(\Delta^p) \otimes C_{n-i}(\Delta^q) \) be
  \[ e_{p,q}^S = \mathbf{1}_{\Delta^p} \otimes \mathbf{1}_{\Delta^q} \]
- \( S_n \) is free with models \( \mathcal{M}_n^S \)
Proof of Eilenberg-Zilber (continued).

- Claim that each $(\Delta^p, \Delta^q) \in M^S_n$ is $T$-acyclic
  - In other words, we must show that for $k > 0$ we have $H_k(T(\Delta^p, \Delta^q)) = 0$

$$
H_k(T(\Delta^p, \Delta^q)) = H_k(C(\Delta^p \times \Delta^q))
= H_k(\Delta^p \times \Delta^q)
\cong H_k(\{\ast\})
= \begin{cases} 
0, & k \neq 0 \\
\mathbb{Z}, & k = 0
\end{cases}
$$

This concludes the proof of the Künneth Formula (for spaces).
More applications of Acyclic Model Theorem

• We could have used Acyclic Model Theorem to prove homotopy invariance of $H$ as follows

  • Let $T : \text{Top} \to \text{Chain}$ be

    $$T(X) = C(X)$$

  • Let $S : \text{Top} \to \text{Chain}$ be

    $$S(X) = C(X \times I)$$

  • $\mathcal{M}_n = \{\Delta^n\}$ with $e_n = 1_{\Delta^n} \in C_n(\Delta^n)$
  • $T_n$ is free with models $\mathcal{M}_n$ (observe definitions)
  • $\Delta^n$ is $S$-acyclic (cone each sing. simplex to fixed $p \in \Delta^n \times I$ to show $H_k(\Delta^n \times I) = 0$ for $k > 0$).
  • Thus there is a unique chain homotopy type of chain maps

    $$\gamma : C(X) \to C(X \times I)$$

    inducing the natural isomorphism $H_0(X) \cong H_0(X \times I)$. 
More applications of Acyclic Model Theorem

- Let $h_0, h_1 : X \to X \times I$ be the maps
  \[ h_0(x) = (x, 0) \quad h_1(x) = (x, 1) \]

- $h_0$ and $h_1$ induce $h_0\# : C(X) \to C(X \times I)$ which in turn induce the natural isomorphism $H_0(X) \cong H_1(X \times I)$
- Hence $h_0\#$ and $h_1\#$ are chain homotopic.
- No suppose $f_0, f_1 : X \to Y$ are homotopic
- Have homotopy $F : X \times I \to Y$ from $f_0$ to $f_1$ with $f_0 = F \circ h_0$ and $f_1 = F \circ h_1$
- Hence
  \[ f_0\# = F_\# \circ h_0\# \simeq F_\# \circ h_1\# = f_1\# \]
Definition 172 (Homotopy for pairs)

Let \((X, A)\) and \((Y, B)\) be topological pairs

\[ f, g : (X, A) \to (Y, B) \]

are homotopic (resp. homotopic rel \(X' \subset X\)) if there is

\[ F : (X \times I, A \times I) \to (Y, B) \]

with \(F(x, 0) = f(x)\) and \(F(x, 1) = g(x)\). (resp. further \(F|_{X'} = 1_{X'}\))

Denote the class of \(f : (X, A) \to (Y, B)\) by \([f]\). (resp. \([f]_{X'}\))

Let

\[ [(X, A); (Y, B)] = \left\{ [f] \left| f : (X, A) \to (Y, B) \right. \right\} \]

and

\[ [(X, A); (Y, B)]_{X'} = \left\{ [f]_{X'} \left| f : (X, A) \to (Y, B) \right. \right\} \]
\[ I^0 = \{0\} \quad \partial I^0 = \emptyset \]
\[ I^n = [0, 1]^n \subset \mathbb{R}^n \]
\[ \partial I^n = \left\{ (x_1, \cdots, x_n) \mid \text{there is } i \text{ s.t. } x_i = 0 \text{ or } x_i = 1 \right\} \]
\[ I^{n-1} = \left\{ (x_1, \cdots, x_n) \mid x_n = 0 \right\} \]
\[ J^{n-1} = \partial I^n \setminus \text{int}(I^{n-1}) \]

**Definition 173 (Homotopy groups)**

\((X, x_0)\) a pointed space and \(n \geq 0\)

\[ \pi_n(X, x_0) = [(I^n, \partial I^n); (X, x_0)] \]

\((X, A, x_0)\) pointed pair of spaces and \(n \geq 1\)

\[ \pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{n-1}); (X, A, x_0)] \]
Definition 174 (Group structure of homotopy groups)

\([f], [g] \in \pi_n(X, x_0)\) with \(n \geq 1\) or \([f], [g] \in \pi_n(X, A, x_0)\) with \(n \geq 2\)

\[
(f \cdot g)(x_1, \cdots, x_n) = \begin{cases} 
  f(2x_1, x_2, \cdots, x_n), & x_1 \leq \frac{1}{2} \\
  g(2x_1 - 1, x_2, \cdots, x_n), & x_1 \geq \frac{1}{2}
\end{cases}
\]

Setting

\([f] \cdot [g] = [f \cdot g]\)

gives \(\pi_n\) a group structure.

- \(\pi_0(X, x_0)\) and \(\pi_1(X, A, x_0)\) are sets with distinguished “identity” element \([c_{x_0}]\)
- \(\pi_1(X, x_0)\) and \(\pi_2(X, A, x_0)\) are nonabelian groups (in general)
- \(\pi_n(X, x_0)\) and \(\pi_{n+1}(X, A, x_0)\) are abelian for \(n \geq 2\)
\[ D^n = \left\{ x \in \mathbb{R}^n \mid |x| \leq 1 \right\} \]
\[ S^{n-1} = \partial D^n = \left\{ x \in \mathbb{R}^n \mid |x| = 1 \right\} \]
\[ p = (1, 0, \cdots, 0) \]

**Definition 175 (Homotopy groups (alt. def.))**

\((X, x_0)\) a pointed space and \(n \geq 0\)
\[ \pi_n(X, x_0) = [(S^n, p); (X, x_0)] \]

\((X, A, x_0)\) pointed pair of spaces and \(n \geq 1\)
\[ \pi_n(X, A, x_0) = [(D^n, S^{n-1}, p); (X, A, x_0)] \]
Theorem 176 (Long exact sequence of homotopy groups)

(\(X, A\)) a pair of spaces. There is a natural long exact sequence

\[
\cdots \rightarrow \pi_{n+1}(X, A, x_0) \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \cdots
\]

\[
\cdots \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{j_*} \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0)
\]

• For last three maps “ker” means the preimage of \([c_{x_0}]\).
• \(i_*\) and \(j_*\) are obvious inclusion maps
• Given \(f : (I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)\)

\[
\partial[f] = [f|_{I^{n-1}}]
\]
Theorem 177 (Long exact homotopy sequence of triple)

\((X, A, B)\) a triple of spaces. There is a natural long exact sequence

\[
\cdots \rightarrow \pi_{n+1}(X, A, x_0) \xrightarrow{\partial} \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \cdots \\
\cdots \xrightarrow{j_*} \pi_2(X, A, x_0) \xrightarrow{\partial} \pi_1(A, B, x_0) \xrightarrow{i_*} \pi_1(X, B, x_0) \xrightarrow{j_*} \pi_1(X, A, x_0)
\]

Are homotopy groups a homology theory?

- Exactness Axiom holds (by above)
- Homotopy Axiom holds
- Dimension Axiom fails (but not badly)
- Additivity Axiom fails
- Excision Axiom fails (big problem)
Definition 178 (n-connectivity)

A space $X$ is $n$-connected if

$$\pi_k(X, x_0) = 0 \quad \text{for all } k \leq n$$

A topological pair $(X, A)$ is $n$-connected if

$$\pi_k(X, A, x_0) = 0 \quad \text{for all } k \leq n$$

- A space is 0-connected iff it is path connected
- A space is 1-connected iff it is simply connected
- $S^n$ is $(n-1)$-connected but not $n$-connected
Let $X$ be a space and

$$G = \pi_1(X, x_0)$$

Then we have a $G$-action on $\pi_n(X, x_0)$

For $n \geq 2$ this makes $\pi_n(X, x_0)$ a $G$-module

**Definition 179 ($G$-module)**

A right or left $G$-module $M$ is a abelian group with a right or left $\mathbb{Z}[G]$-module structure.

**Definition 180 ($G$-coinvaraiants)**

Given a $G$-module $M$, the $G$-coinvaraiants of $M$ are

$$M_G = \frac{M}{\langle gm - m \mid g \in G, m \in M \rangle}$$

or equivalently

$$M_G = M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$$

where $\mathbb{Z}$ is the trivial $G$-module.
The Hurewicz Homomorphism

- Basic idea of Hurewicz Homomorphism
- Given $[\alpha] \in \pi_n(X)$
- want $h([\alpha]) \in H_n(X)$
- have map $\alpha : (S^n, p) \to (X, X_0)$
- Choose a generator $z_n \in H_n(S^n, p)$
- define $h([\alpha]) = \alpha_* z_n$
- To get naturality of $h$ must choose $z_n$ compatibly for all $n$
- Some work to show $h$ is a homomorphism.
• Goal: define Hurewicz homomorphism
• Must inductively choose generators of
  \[ H_n(I^n, \partial I^n) \]
  for all \( n \).
• Choose \( z_1 \in H_1(I, \partial I) \) to be unique element such that
  \[ \partial z_1 = [1] - [0] \in H_0(\partial I = \{0, 1\}) \]
• By Excision have an isomorphism
  \[ j : H_{n-1}(I^{n-1}, \partial I^{n-1}) \cong H_{n-1}(\partial I^n, J^{n-1}) \]
• Given \( z_{n-1} \in H_{n-1}(I^{n-1}, \partial I^{n-1}) \) choose \( z_n \in H_n(I^n, \partial I^n) \) so that
  \[ \partial z_n = jz_{n-1} \]
Definition 181 (The Hurewicz Homomorphism)

The Hurewicz map is the function

\[ h : \pi_n(X, A, x_0) \to H_n(X, A) \]

where given

\[ [\alpha] \in \pi_n(X, A, x_0) \]

we let

\[ h([\alpha]) = \alpha_*z_n \in H_n(X, A) \]

Letting \( A = x_0 \) and assuming \( n \geq 1 \) we get

\[ h : \pi_n(X, A, x_0) \to H_n(X, x_0) \cong \tilde{H}_n(X) \cong H_n(X) \]

- Claim: \( h \) is well-defined
- Given \( \alpha, \beta : (I^n, \partial I^n) \to (X, A) \)
- if \( \alpha \simeq \beta \)
- then by Homotopy Axiom \( \alpha_* = \beta_* \)
- So \( \alpha_*z_n = \beta_*z_n \)
Theorem 182 (The Hurewicz Homomorphism)

For $(X, A)$ a topological pair and $n \geq 1$ the Hurewicz map

$$h : \pi_n(X, A, x_0) \to H(X, A)$$

defined above satisfies the following:

1. $h$ is a homomorphism.
2. $h$ is natural in that we have commutative

$$
\begin{array}{ccc}
\cdots & i_* & \pi_2(X, x_0) \\
& j_* & \downarrow h \\
\pi_2(X, A, x_0) & \partial & \pi_1(A, x_0) \\
& i_* & \downarrow h \\
\pi_1(X, x_0) & \partial & \pi_1(X, x_0) \\
& i_* & \downarrow h \\
\cdots & i_* & H_2(X) \\
& j_* & \downarrow h \\
H_2(X, A) & \partial & H_1(A) \\
& i_* & \downarrow h \\
H_1(X) & \partial & H_1(X)
\end{array}
$$

3. $h$ is also natural in that for $f : (X, A, x_0) \to (Y, B, y_0)$ we have commutative

$$
\begin{array}{ccc}
\pi_n(X, A, x_0) & f_* & \pi_n(Y, B, y_0) \\
& h & \downarrow h \\
H_n(X, A) & f_* & H_n(Y, B)
\end{array}
$$
Claim: Given \( f : (X, A, x_0) \to (Y, B, y_0) \) we have commutative

\[
\begin{array}{ccc}
\pi_n(X, A, x_0) & \xrightarrow{f_*} & \pi_n(Y, B, y_0) \\
\downarrow h & & \downarrow h \\
H_n(X, A) & \xrightarrow{f_*} & H_n(Y, B)
\end{array}
\]

- Given \( \alpha : (I^n, \partial I^n) \to (X, A) \)

\[
h(f_*[\alpha]) = h([f\alpha]) = (f\alpha)_*z_n = f_* (\alpha_*z_n) = f_* (h([\alpha]))
\]

Note: We have established commutativity of many squares of LES
Proof of Theorem 182 (continued).

Claim: We have commutative

\[
\begin{array}{ccl}
\pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \\
\downarrow h & & \downarrow h \\
H_n(X, A) \xrightarrow{\partial} H_{n-1}(A)
\end{array}
\]

- Given \( \alpha : (I^n, \partial I^n) \to (X, A) \)
- Let \( \alpha' : (I^{n-1}, \partial I^{n-1}) \to (A, x_0) \) satisfy \( \alpha' = \alpha|_{I^{n-1}} \)
- Then \( \partial[\alpha] = [\alpha'] \)
- If \( j : (I^{n-1}, \partial I^{n-1}) \to (\partial I^n, J^{n-1}) \) is inclusion we have \( \alpha' = \alpha|_{\partial I^n} \circ j \)

\[
h(\partial[\alpha]) = h([\alpha']) = \alpha'_*z_{n-1} = (\alpha|_{\partial I^n})_*j_*z_{n-1} = (\alpha|_{\partial I^n})_*\partial z_n = \alpha_*\partial z_n = \partial\alpha_*z_n = \partial h([\alpha])
\]
Proof of Theorem 182 (continued).

1. Claim: \( h : \pi_n(X, A, x_0) \to H_n(X, A) \) is a homomorphism

   - Let \( L \subset I^n \) be \( \{(x_1, \cdots, x_n) \mid x_1 \leq \frac{1}{2}\} \)
   - Let \( U \subset I^n \) be \( \{(x_1, \cdots, x_n) \mid x_1 \geq \frac{1}{2}\} \)
   - Let \( \partial L = L - \text{int} L \) and \( \partial U = U - \text{int} U \)
   - Consider inclusions

\[
i_L : (L, \partial L) \to (I^n, \partial L \cup \partial U) \\
i_U : (U, \partial U) \to (I^n, \partial L \cup \partial U)
\]

   - Get isomorphism

\[
i_{L*} \oplus i_{U*} : H_n(L, \partial L) \oplus H_n(U, \partial U) \to H_n(I^n, \partial L \cup \partial U)
\]

   - Consider “squeeze” maps

\[
s_L : (I^n, \partial I^n) \to (L, \partial L) \qquad s_U : (I^n, \partial I^n) \to (U, \partial U)
\]

defined by

\[
s_L(x_1, \cdots, x_n) = \left(\frac{x_1}{2}, \cdots, x_n\right) \quad s_U(x_1, \cdots, x_n) = \left(\frac{x_1+1}{2}, \cdots, x_n\right)
\]
Proof of Theorem 182 (continued).

1. \( h : \pi_n(X, A, x_0) \to H_n(X, A) \) is a homomorphism (continued)
   
   - We have inclusion
     
     \[ i : (I^n, \partial I^n) \to (I^n, \partial L \cup \partial U) \]

   - For \( z_n \in H_n(I^n, \partial I^n) \)
     
     \[ i_* z_n = i_* s_* L_* z_n + i_* s_* U_* z_n \]

   - Now given \( \alpha, \beta : (I^n, \partial I^n) \to (X, A) \)
     
     \[ h([\alpha] \cdot [\beta]) = h([\alpha \cdot \beta]) = (\alpha \cdot \beta)_* z_n = ((\alpha \cdot \beta) \circ i)_* z_n = (\alpha \cdot \beta)_* i_* z_n = (\alpha \cdot \beta)_*(i_* s_* L_* + i_* s_* U_*) z_n = \alpha z_n + \beta z_n = h([\alpha]) + h([\beta]) \]
### Theorem 183

Let $G = \pi_1(X, x_0)$ then for $n \geq 1$

$$h : \pi_n(X, x_0) \to H_n(X)$$

factors through the quotient map

$$q : \pi_n(X, x_0) \to \pi_n(X, x_0)_G$$

to $\pi_n(X, x_0)_G$ the $G$-coinvariants of $\pi_n(X, x_0)$

### Theorem 184

$(X, A)$ a topological pair. Let $G = \pi_1(A, x_0)$ then for $n \geq 2$

$$h : \pi_n(X, A, x_0) \to H_n(X)$$

factors through the quotient map

$$q : \pi_n(X, A, x_0) \to \pi_n(X, A, x_0)_G$$

to $\pi_n(X, A, x_0)_G$ the $G$-coinvariants of $\pi_n(X, A, x_0)$
Proof of Theorems 183 and 184.

- \([f] \in \pi_1(X, x_0)\)
- \([\alpha] \in \pi_n(X, x_0)\)
- Standard argument gives explicit homotopy between \(\alpha\) and \(f \cdot \alpha\)
Exercise 185 (HW9 - Problem 1)

Decide whether or not for all spaces $X$ and all $n \geq 1$ the Hurewicz homomorphism

$$h : \pi_n(X, x_0) \to H_n(X)$$

is surjective.
Exercise 186 (HW9 - Problem 2)

Hatcher pg. 389-392: 10, 11, 19, 20

You may use any results from Section 4.2 for these problems including the Hurewicz Theorem and Whitehead’s Theorem