Lecture 29 - 3/7/2011

Lecture 30 - 3/8/2011

Lecture 31 - 3/9/2011

Math 757
Homology theory

March 9, 2011

Theorem 183

Let $G = \pi_1(X, x_0)$ *then for* $n \geq 1$

$$h : \pi_n(X, x_0) \to H_n(X)$$

factors through the quotient map

$$q : \pi_n(X, x_0) \to \pi_n(X, x_0)_G$$

to $\pi_n(X, x_0)_G$ *the* G*-coinvariants of* $\pi_n(X, x_0)$

Theorem 184

(X, A) *a topological pair. Let* $G = \pi_1(A, x_0)$ *then for* $n \geq 2$

$$h : \pi_n(X, A, x_0) \to H_n(X)$$

factors through the quotient map

$$q : \pi_n(X, A, x_0) \to \pi_n(X, A, x_0)_G$$

to $\pi_n(X, A, x_0)_G$ *the* G*-coinvariants of* $\pi_n(X, A, x_0)$

Proof of Theorems 183 and 184.

- $[f] \in \pi_1(X, x_0)$
- $[\alpha] \in \pi_n(X, x_0)$
- Standard argument gives explicit homotopy between α and $f \cdot \alpha$

$\square$

Exercise 185 (HW9 - Problem 1)

Decide whether or not for all spaces X and all $n \geq 1$ the Hurewicz homomorphism

$$h : \pi_n(X, x_0) \to H_n(X)$$

is surjective.

Exercise 186 (HW9 - Problem 2)

Hatcher pg. 389-392: 10, 11, 19, 20

You may use any results from Section 4.2 for these problems including the Hurewicz Theorem and Whitehead's Theorem

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Definition 187

Let

$$C_n(X, A)^k_{x_0} \subset C_n(X, A)$$

Be linear combinations of maps

$$\sigma : \Delta^n \to X$$

Where σ takes vertices of Δ^n to x_0 and the k-skeleton of Δ^n to A.

Goal: Prove that if X is path-connected and (X, A) is k-connected for some $k \geq 0$ then the inclusion

$$C(X, A)^k_{x_0} \subset C(X, A)$$

is a chain homotopy equivalence.

Lemma 188

Suppose X is a space and $D \subset C(X)$ is a sub-chain complex satisfying

1. *D is generated by singular simplices*
2. *For each singular simplex $\sigma : \Delta^n \to X$ there is*

$$P_\sigma : \Delta^n \times I \to X$$

3. $P_\sigma(z, 0) = \sigma(z)$
4. *If $\bar{\sigma} : \Delta^n \times I \to X$ is given by $\bar{\sigma}(x) = P_\sigma(z, 1)$ then $\bar{\sigma} \in D$*
5. $P_{\partial^i \sigma} = P_\sigma | \partial^i \sigma \times I$

Then the inclusion $D \subset C(X)$ is a chain homotopy equivalence.

Proof of Lemma 188.

- Let
$$i : D \to C(X)$$
be the inclusion.
- Let
$$p : C(X) \to D$$
be given by $p(\sigma) = \bar{\sigma}$
- p is a chain map since
$$\begin{aligned} p(\partial\sigma) &= \sum_j (-1)^j P_\sigma | \partial^j \sigma \times \{1\} \\ &= \partial\bar{\sigma} \\ &= \partial p(\sigma) \end{aligned}$$
- $p \circ i = \mathbf{1}_D$
- Claim: $i \circ p \simeq \mathbf{1}_{C(X)}$

Proof of Lemma 188 *(continued)*.

- Claim: $i \circ p \simeq \mathbf{1}_{C(X)}$
  - Let
$$h_0, h_1 : \Delta^n \to \Delta^n \times I$$
be the maps
$$h_0(z) = (z, 0) \qquad h_1(z) = (z, 1)$$
  - We showed there is a chain homotopy $T : C(\Delta^n) \to C(\Delta^n \times I)$ between $h_{0\sharp}$ and $h_{1\sharp}$.
  - Let $T_X : C(X) \to C(X)$ be
$$T_X(\sigma) = (P_{\sigma\sharp} T)(\mathbf{1}_{\Delta^n})$$
  - Then
$$\begin{aligned}(\partial T_X + T_X \partial)(\sigma) &= \partial(P_{\sigma\sharp} T)(\mathbf{1}_{\Delta^n}) + (P_{\sigma\sharp} T)(\partial \mathbf{1}_{\Delta^n}) \\ &= P_{\sigma\sharp} \partial T(\mathbf{1}_{\Delta^n}) + P_{\sigma\sharp} T \partial(\mathbf{1}_{\Delta^n}) \\ &= P_{\sigma\sharp}(\partial T + T\partial)(\mathbf{1}_{\Delta^n}) \\ &= P_{\sigma\sharp}(h_{1\sharp} - h_{0\sharp})(\mathbf{1}_{\Delta^n}) \\ &= \bar{\sigma} - \sigma \\ &= (i \circ p - \mathbf{1}_{C(X)})(\sigma)\end{aligned}$$
$\square$

Proposition 189

1. *X is path connected*
2. *For some $k \geq 0$ we have (X, A) is k-connected ($\pi_n(X, A, x_0) = 0$ for $n \leq k$)*

Then the inclusion $C(X, A)^k_{x_0} \subset C(X)$ is a chain homotopy equivalence.

Proof of Proposition 189.

- Given Lemma 188 above want $P_\sigma : \Delta^n \times I \to X$ for each $\sigma \in \mathcal{S}_n(X)$
- Will define P inducting on n
- Suppose $n = 0$ then $\Delta^0 = \{1\}$
  - Each $\sigma : \Delta^0 \to X$ is given by a point $\sigma(1) \in X$
  - For each $x \in X$ choose path $\gamma_x : I \to X$ with $\gamma_x(0) = x$ and $\gamma_x(1) = x_0$
  - For $x_0 \in X$ choose $\gamma_{x_0} = c_{x_0}$
  - Let $P_\sigma : \Delta^0 \times I \to X$ be

$$P_\sigma(1, t) = \gamma_{\sigma(1)}(t)$$

  - We have P_σ defined for all $\sigma \in \mathcal{S}_0(X)$
- Suppose $0 < n \leq k$ and P_σ defined for all $\sigma \in \mathcal{S}_m(X)$ with $m < n$.
  - Given $\sigma \in \mathcal{S}_n(X)$
  - If $\sigma \in C(X, A)^k_{x_0}$ let $P_\sigma : \Delta^n \times I \to X$ be

$$P_\sigma(z, t) = \sigma(z)$$

Proof of Proposition 189 *(continued)*.

- If $\sigma \notin C(X,A)^k_{x_0}$
- Define P_σ on $\Delta^n \times \{0\}$

$$P_\sigma(z,0) = \sigma(z)$$

- On $\partial\Delta^n \times I$ inductive assumption gives def of P_σ
- $\pi_n(X,A,x_0) = 0$ so there is a homotopy which we use to fill in P_σ such that
- $P\sigma(\Delta^n \times 1) \subset A$

- Suppose $n > k$ and P_σ defined for all $\sigma \in \mathcal{S}_m(X)$ with $m < n$.
  - Given $\sigma \in \mathcal{S}_n(X)$
  - k-skeleton of σ contained in union of faces of σ
  - so σ is in $C(X,A)^k_{x_0}$ iff every face of σ is in $C(X,A)^k_{x_0}$
  - By inductive assumption P_σ is defined on

$$\partial\Delta^n \times I \cup \Delta^n \times \{0\}$$

  - "Fill in" P_σ for $\Delta^n \times \{i\}$ with map that follows $\partial\Delta^n \times [i,0] \cup \Delta^n \times \{0\}$
  - Faces of $\bar{\sigma}$ are in $C(X,A)^k_{x_0}$ by inductive assumption so $\bar{\sigma}$ is.

Definition 190

$$H_n^{(k,x_0)}(X,A) = H_n\left(\frac{C(X,A)_{x_0}^k}{C(X,A)_{x_0}^k \cap C(A)}\right)$$

- Inclusion of chain complexes induce homomophisms

$$\cdots \to H_n^{(2,x_0)}(X,A) \to H_n^{(1,x_0)}(X,A) \to H_n^{(0,x_0)}(X,A) \to H_n(X,A)$$

Corollary 191

1. *A is path connected*
2. *(X,A) k-connected*

Then for all n above homomorphisms induce isomorphisms

$$H_n^{(k,x_0)}(X,A) \cong H_n(X,A)$$

Proof of Corollary 191.

- Claim 1: $C(X,A)^k_{x_0} \cap C(A) \subset C(A)$ is a chain homotopy equivalence
  - A is path-connected so $(A, \{x_0\})$ is 0-connected
  - By Proposition 189 we get above chain homotopy equivalence.
- Claim 2: $C(X,A)^k_{x_0} \subset C(X)$ is a chain homotopy equivalence
  - (X,A) is k-connected and X path-connected
  - By Proposition 189 we get above chain homotopy equivalence.
- Have morphism of short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C(X,A)^k_{x_0} \cap C(A) \arrow[r] \arrow[d] & C(X,A)^k_{x_0} \arrow[r] \arrow[d] & \frac{C(X,A)^k_{x_0}}{C(X,A)^k_{x_0} \cap C(A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(A) \arrow[r] & C(X) \arrow[r] & C(X,A) \arrow[r] & 0
\end{tikzcd}$$

- induces morphism of long exact sequence with 2/3 maps isomorphisms
- 5-Lemma gives desired isomorphism

$\square$

- X and A path-connected
- $G = \pi_1(A, x_0)$
- (X, A) n-connected
- For now let $\pi_n(X, A, x_0)$ be homotopy classes of maps

$$\alpha : (\Delta^n, \partial\Delta^n, \partial^0\Delta^n) \to (X, A, x_0)$$

- Any such map α is a singular simplex in $C(X, A)_{x_0}^{(n-1)}$ so the Hurewicz homomorphism

$$h : \pi_n(X, A, x_0) \to H_n(X, A)$$

factors as

$$\pi_n(X, A, x_0) \xrightarrow{q} \pi_n(X, A, x_0)_G \xrightarrow{h''} H_n^{(n-1, x_0)}(X, A) \to H_n(X, A)$$

- Let $h' : \pi_n(X, A, x_0)_G \to H_n(X, A)$ be the last composition.

Theorem 192 (RHI(n) Relative Hurewicz Isomorphism)

($n \geq 2$). Let A be path connected, (X, A) be $(n-1)$-connected and let $G = \pi_1(A, x_0)$. Then

$$h' : \pi_n(X, A, x_0)_G \to H_n(X, A)$$

is an isomorphism.

Theorem 193 (AHI(n) Hurewicz Isomorphism)

($n \geq 1$). Let X be $(n-1)$-connected. Then

$$h : \pi_n(X, x_0) \to H_n(X, x_0)$$

is an isomorphism.

- In the $(n+1)$-simplex Δ^{n+1} with $(n-1)$-skeleton $(\Delta^{n+1})^{n-1}$
- Let $\lambda^i_{n+1} = [e_0, \cdots, \widehat{e_i}, \cdots, e_n]$

$$\lambda^0_{n+1} : (\Delta^n, \partial\Delta^n, e_0) \to (\partial\Delta^{n+1}, (\Delta^{n+1})^{n-1}, e_1)$$

$$\lambda^i_{n+1} : (\Delta^n, \partial\Delta^n, e_0) \to (\partial\Delta^{n+1}, (\Delta^{n+1})^{n-1}, e_0)$$

- Let $\gamma : I \to \Delta^{n+1}$ be the linear path with $\gamma(t) = (1-t)e_0 + te_1$.
- Let $b_n \in \pi_n(\partial\Delta^{n+1}, (\Delta^{n+1})^{n-1}, e_0)$

$$b_2 = \gamma \cdot [\lambda^0_3][\lambda^2_3][\lambda^1_3]^{-1}[\lambda^3_3]^{-1}$$

$$b_n = \gamma \cdot [\lambda^0_{n+1}] + \sum_i (-1)^i [\lambda^i_{n+1}]$$

- We have the inclusion

$$j : (\partial\Delta^{n+1}, (\Delta^{n+1})^{n-1}, e_0) \to (\Delta^{n+1}, (\Delta^{n+1})^{n-1}, e_0)$$

Lemma 194 (SF(n) Homotopy sum of simplicial faces)

$(n \geq 1)$.

$$j_\star b_n = 0$$

Proof of RHI(n), AHI(n) and SF(n).

The proof will follow these steps

1. SF(1) and AHI(1) were proved when we showed $\pi_1(X)_{\mathrm{Ab}} \cong H_1(X)$
2. AHI(1), AHI(2), $\cdots$, AHI($n-1$) $\implies$ SF(n) for $n \geq 2$
3. SF(n) $\implies$ RHI(n) for $n \geq 2$
4. RHI(n) $\implies$ AHI(n) for $n \geq 2$

Part 1 is shown so we start with part 2

2. Claim: AHI(1), AHI(2), $\cdots$, AHI($n-1$) $\implies$ SF(n) for $n \geq 2$
   - Consider the homotopy long exact sequence of the triple $(\Delta^{n+1}, \partial\Delta^{n+1}, (\Delta^{n+1})^{n-1})$ and the pair $(\partial\Delta^{n+1}, (\Delta^{n+1})^{n-1})$

$$\begin{tikzcd}
\pi_{n+1}(\Delta^{n+1}, \partial\Delta^{n+1}, e_0) \arrow[dr, "\partial'"] \arrow[dd, "\partial"'] & & \pi_n(\Delta^{n+1}, (\Delta^{n+1})^{n-1}, e_0) \\
& \pi_n(\partial\Delta^{n+1}, (\Delta^{n+1})^{n-1}, e_0) \arrow[ur, "j_\star"] \arrow[dr, "\partial''"] & \\
\pi_n(\partial\Delta^{n+1}, e_0) \arrow[ur, "i_\star"] & & \pi_{n-1}((\Delta^{n+1})^{n-1}, e_0)
\end{tikzcd}$$

   - Commutativity comes from naturality of homotopy LES and inclusion $(\Delta^{n+1}, \partial\Delta^{n+1}, e_0) \subset (\Delta^{n+1}, \partial\Delta^{n+1}, (\Delta^{n+1})^{n-1})$

Proof of RHI(n), AHI(n) and SF(n) *(continued)*.

2. Claim: AHI(1), AHI(2), $\cdots$, AHI($n-1$) $\implies$ SF(n) for $n \geq 2$

- $$\begin{tikzcd}
\pi_{n+1}(\Delta^{n+1}, \partial\Delta^{n+1}, e_0) \arrow[dr, "\partial'"] \arrow[dd, "\partial"', "\cong"] & & \pi_n(\Delta^{n+1}, (\Delta^{n+1})^{n-1}, e_0) \\
& \pi_n(\partial\Delta^{n+1}, (\Delta^{n+1})^{n-1}, e_0) \arrow[ur, "j_\star"] \arrow[dr, "\partial''"] & \\
\pi_n(\partial\Delta^{n+1}, e_0) \arrow[ur, "i_\star"] & & \pi_{n-1}((\Delta^{n+1})^{n-1}, e_0)
\end{tikzcd}$$

- Δ^{n+1} is contractible so in homotopy LES of pair $(\Delta^{n+1}, \partial\Delta^{n+1})$ we see ∂ is an isomorphism.
- Hence

$$\begin{aligned}
\ker j_\star &= \operatorname{Im} \partial' \\
&= \operatorname{Im}(i_\star \partial) \\
&= \operatorname{Im} i_\star \\
&= \ker \partial''
\end{aligned}$$

- Hence claim SF(n) ($j_\star b_n = 0$) will follow from showing $\partial''(b_n) = 0$

Proof of RHI(n), AHI(n) and SF(n) *(continued)*.

2. Claim: AHI(1), AHI(2), $\cdots$, AHI($n-1$) $\implies$ SF(n) for $n \geq 2$

- $$\begin{tikzcd}
\pi_{n+1}(\Delta^{n+1}, \partial\Delta^{n+1}, e_0) \arrow[dr, "\partial'"] \arrow[dd, "\partial"', "\cong"] & & \pi_n(\Delta^{n+1}, (\Delta^{n+1})^{n-1}, e_0) \\
& \pi_n(\partial\Delta^{n+1}, (\Delta^{n+1})^{n-1}, e_0) \arrow[ur, "j_\star"] \arrow[dr, "\partial''"] & \\
\pi_n(\partial\Delta^{n+1}, e_0) \arrow[ur, "i_\star"] & & \pi_{n-1}((\Delta^{n+1})^{n-1}, e_0)
\end{tikzcd}$$

- If $n = 2$ easy calculation shows $\partial'' b_2 = 1 \in \pi_1((\Delta^3)^1, e_0) \cong F_3$
- If $n \geq 3$ then $(\Delta^{n+1})^{n-1}$ contains 2-skeleton of Δ^n so $(\Delta^{n+1})^{n-1}$ has same trival π_1 as Δ^n.
- For $k \leq n-2$

$$H_k((\Delta^n)^{n-1}, v_0) \cong H_k(\Delta^n, v_0) = 0$$

- By AHI(1), $\cdots$, AHI($n-2$) we get $(\Delta^{n+1})^{n-1}$ is $(n-2)$-connected.
- By AHI($n-1$) we have an isomorphism

$$h : \pi_{n-1}((\Delta^{n+1})^{n-1}, v_0) \cong H_{n-1}(\Delta^{n+1})^{n-1}, v_0)$$

Proof of RHI(n), AHI(n) and SF(n) *(continued)*.

2. Claim: AHI(1), AHI(2), $\cdots$, AHI($n-1$) $\implies$ SF(n) for $n \geq 2$

- $$\begin{tikzcd}
\pi_{n+1}(\Delta^{n+1}, \partial\Delta^{n+1}, e_0) \arrow[dr, "\partial'"] \arrow[dd, "\partial"', "\cong"] & & \pi_n(\Delta^{n+1}, (\Delta^{n+1})^{n-1}, e_0) \\
& \pi_n(\partial\Delta^{n+1}, (\Delta^{n+1})^{n-1}, e_0) \arrow[ur, "j_\star"] \arrow[dr, "\partial''"] & \\
\pi_n(\partial\Delta^{n+1}, e_0) \arrow[ur, "i_\star"] & & \pi_{n-1}((\Delta^{n+1})^{n-1}, e_0)
\end{tikzcd}$$

- In $\pi_{n-1}((\Delta^{n+1})^{n-1}, v_0)$ computation is difficult but in $H_{n-1}((\Delta^{n+1})^{n-1}, v_0)$

- $$\begin{aligned}
h\partial'' b_n &= \partial'' h(b_n) \\
&= \partial'' \left[\sum (-1)^i \lambda^i_{n+1}\right] \\
&= \partial\partial\left[\mathbf{1}_{\Delta^{n+1}}\right] \\
\text{so} \quad \partial'' b_n &= 0
\end{aligned}$$

- Part 2 is complete.

Proof of RHI(n), AHI(n) and SF(n) *(continued)*.

3. Claim: SF(n) $\implies$ RHI(n) for $n \geq 2$
   - The Hurewicz homomorphism
   $$h : \pi_n(X, A, x_0) \to H_n(X, A)$$
   - factors as
   $$\pi_n(X, A, x_0)_G \xrightarrow{h''} H_n^{(n-1,x_0)}(X, A) \xrightarrow{\cong} H_n(X, A)$$
   - Enough to construct inverse $w : H_n^{(n-1,x_0)}(X, A) \to \pi_n(X, A, x_0)_G$ of h''
   - First define w on chains $C_n(X, A)_{x_0}^{n-1}$
   - If $\sigma : (\Delta^n, \partial\Delta^n, e_0) \to (X, A, x_0)$ is a sing. simpl. in $C_n(X, A)_{x_0}^{n-1}$
   - then $[\sigma] \in \pi_n(X, A, x_0)_G$
   - Set $w([\sigma]) = [\sigma]$. By freeness of $C_n(X, A)_{x_0}^{n-1}$ have a homorphism
   $$w : C_n(X, A)_{x_0}^{n-1} \to \pi_n(X, A, x_0)_G$$

Proof of RHI(n), AHI(n) and SF(n) *(continued)*.

3. Claim: SF(n) $\implies$ RHI(n) for $n \geq 2$
   - If $\operatorname{Im}\sigma \subset A$ then $w[\sigma] = 0$. So w induces a homorphism
   $$w : \frac{C_n(X,A)^{n-1}_{x_0}}{C_n(X,A)^{n-1}_{x_0} \cap C_n(A)} \to \pi_n(X,A,x_0)_G$$
   - If $w\partial = 0$ then w induces map on homology.
   - Given
   $$\sigma : (\Delta^{n+1}, \partial\Delta^{n+1}, e_0) \to (X, A, x_0)$$
   - Apply SF(n) to get
   $$\begin{aligned} w\partial\sigma &= \sum(-1)^i[\partial^i\sigma] \\ &= \sigma_\star j_\star \sum(-1)^i[\lambda^i_{n+1}] \\ &= \sigma_\star j_\star b_n \\ &= 0 \end{aligned}$$
   - $w \circ h'' = \mathbf{1}$ so h is an isomorphism.
   - Part 3 is complete.

Proof of RHI(n), AHI(n) and SF(n) *(continued)*.

4. Claim: RHI(n) $\implies$ AHI(n) for $n \geq 2$
   - If X is $(n-1)$-connected then the pair (X, x_0) is $(n-1)$-connected.
   - Apply RHI(n) to the pair (X, x_0).
   - Get AHI(n).
   - This completes the proof.

$\square$