1. Find the slope of the line tangent to the curve \( r = 1 + 2 \sin(2\theta) \) at the point \((3, \pi/4)\).

Then find an equation for this line.

**Background:** Recall that to find the slope of a tangent line to an equation in polar coordinates, we convert the polar equation to a parametric equation. Then we use the techniques of parametric equations to find the derivative of the curve. Recall that we change between polar coordinates and Cartesian coordinates through the equations

\[
x = r \cos \theta \\
y = r \sin \theta.
\]

Thus if we have a polar function \( r = f(\theta) \), then we can derive the parametric equation:

\[
\theta \mapsto (f(\theta) \cos \theta, f(\theta) \sin \theta).
\]

From here we can find the slope of the tangent line as a function of \( \theta \).

\[
\frac{dy}{dx}(\theta) = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}.
\]

**Solution:** Considering the above discussion we need to find the derivative of \( r(\theta) = 1 + 2 \sin(2\theta) \) with respect to \( \theta \) and then plug everything into the formula we derived. A quick calculation shows:

\[
r'(\theta) = 4 \cos(2\theta).
\]

Thus

\[
\frac{dy}{dx}(\theta) = \frac{(1 + 2 \sin(2\theta)) \cos \theta + (4 \cos(2\theta)) \sin \theta}{-(1 + 2 \sin(2\theta)) \sin \theta + (4 \cos(2\theta)) \cos \theta}.
\]

At the point \((3, \pi/4)\),

\[
\frac{dy}{dx} \left( \frac{\pi}{4} \right) = \frac{(1 + 2 \sin(2 \cdot \frac{\pi}{4})) \cos \frac{\pi}{4} + (4 \cos(2 \cdot \frac{\pi}{4})) \sin \frac{\pi}{4}}{(1 + 2 \sin(2 \cdot \frac{\pi}{4})) \sin \frac{\pi}{4} - (4 \cos(2 \cdot \frac{\pi}{4})) \cos \frac{\pi}{4}}
\]

\[
= \frac{(1 + 2 \cdot 1) \cdot \frac{\sqrt{2}}{2} + (4 \cdot 0) \cdot \frac{\sqrt{2}}{2}}{-(1 + 2 \cdot 1) \cdot \frac{\sqrt{2}}{2} + (4 \cdot 0) \cdot \frac{\sqrt{2}}{2}} = -1
\]

2. Find the points at which \( r = 6 + 3 \cos \theta \) has a horizontal or vertical tangent line.

**Solution:** Recall that for a function polar function \( f(\theta) \),

\[
\frac{dy}{dx}(\theta) = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}.
\]

In this case \( r = f(\theta) \) and \( r'(\theta) = -3 \sin \theta \). Thus

\[
\frac{dy}{dx}(\theta) = \frac{(6 + 3 \cos \theta) \cos \theta - 3 \sin \theta \sin \theta}{-(6 + 3 \cos \theta) \sin \theta - 3 \sin \theta \cos \theta}.
\]

\( r(\theta) \) will have a horizontal tangent when \( \frac{dy}{dx}(\theta) = 0 \), which will occur only when

\[
(6 + 3 \cos \theta) \cos \theta - 3 \sin \theta \sin \theta = 0.
\]

By replacing \( \sin^2 \theta \) with \( 1 - \cos^2 \theta \) we have a the following quadratic equation in terms of \( \cos \theta \).

\[
6 \cos \theta + 3 \cos^2 \theta - 3(1 - \cos^2 \theta) = 0.
\]

Equivalently

\[
-3 + 6 \cos \theta + 6 \cos^2 \theta = 0,
\]

\[
-1 + 2 \cos \theta + 2 \cos^2 \theta = 0.
\]

The roots of the quadratic \( 2x^2 + 2x - 1 \) are

\[
\frac{-2 \pm \sqrt{4 - 4(2)(-1)}}{2 \cdot 2} = \frac{\pm \sqrt{3} - 1}{2}.
\]

Since \( \frac{-\sqrt{3} - 1}{2} < -1 \), only \( \theta \) such that \( \cos \theta = \frac{-\sqrt{3} - 1}{2} \) can be solutions. In the interval \([0, 2\pi]\), the only such \( \theta \) are \( \cos^{-1}(\frac{-\sqrt{3} - 1}{2}) \) and \( 2\pi - \cos^{-1}(\frac{-\sqrt{3} - 1}{2}) \).

We will have a vertical tangent line when the denominator of our derivative is 0. I.e. when

\[
0 = -(6 + 3 \cos \theta) \sin \theta - 3 \sin \theta \cos \theta
\]

\[
= -6 \sin \theta(1 + \cos \theta).
\]

Thus \( \sin \theta = 0 \) or \( \cos \theta = -1 \), and \( \theta = 0, \pi \).
3. Make a sketch of the limacon \( r = 3 + 6 \cos \theta \) and find the area of the inner region.

**Solution:** To begin, let us identify the angles corresponding to the inner region. The graph transitions between inner loop and outer loop when \( 0 = 3 + 6 \cos(\theta) \). In other words for

\[ \theta = \cos^{-1}(3/6) = \frac{2\pi}{3}, \frac{4\pi}{3}. \]

Hence

\[ \text{Area} = \frac{1}{2} \int_{\frac{4\pi}{3}}^{\frac{2\pi}{3}} (3 + 6 \cos \theta)^2 \, d\theta. \]

By symmetry

\[ A = 6 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2} (\cos 3\theta)^2 \, d\theta \]

Furthermore, as our function is of the form \( r = \cos(m\theta) \) a leaf will be centered on the \( x \)-axis. All leaves will be equally spaced, so the other two leaves will be centered on \( 2\pi/3 \) and \( 4\pi/3 \). Let us find the angle bounds associated to the leaf on the \( x \)-axis.

\[ 0 = \cos 3\theta. \]

\[ 3\theta = \frac{\pi}{2} + k\pi \]

for \( k \in \mathbb{N} \).

\[ \theta = \frac{\pi}{6} + \frac{k\pi}{3} \]

for \( k \in \mathbb{N} \). Thus the leaf on the \( x \)-axis ranges from an angle of \(-\pi/6\) to \(\pi/6\). We will find the area of the top half of this leaf and multiply by 6 to find the area of all leaves.

\[ A = 6 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2} (\cos 3\theta)^2 \, d\theta \]

4. Make a sketch of the rose \( r = \cos(3\theta) \) and find the area of the area within all the leaves.

**Solution:** As the coefficient in front of theta is odd, our rose will have 3 leaves.
5. Find the area of the region that lies within the curve \( r = 3 \sin \theta \) and \( r = 3 \cos \theta \).

**Solution:** Recall that each of these graphs is a circle, one centered on the \( x \)-axis and the other on the \( y \)-axis. Here is a quick picture.

Let us calculate the intersection points. Note that each curve intersects the origin, but at different angles. \( 3 \sin \theta = 0 \) at \( \pi/2 \) and \( 3 \cos \theta = 0 \) at 0. By equating the curves and solving for \( \theta \), we can find the second intersection point.

\[
3 \sin \theta = 3 \cos \theta
\]

\[
\tan \theta = 1
\]

\[
\theta = \pi/4.
\]

Our region can be subdivided into two subregions one from the angle 0 to \( \pi/4 \) and the second from a range of \( \pi/4 \) to \( \pi/2 \). The area of the first region is the area under \( \sqrt{3} \sin \theta \). Notice that each has one top function and no bottom function. The first has a top function of \( r = 3 \sin \theta \) and the second has a top function of \( r = \sqrt{3} \cos \theta \).

\[
A_1 = \int_0^{\pi/4} 3 \sin \theta \, d\theta
\]

\[
= -3 \cos \theta \bigg|_0^{\pi/4}
\]

\[
= -3 \left( \frac{\sqrt{2}}{2} - 1 \right).
\]

\[
A_2 = \int_{\pi/4}^{\pi/2} 3 \cos \theta \, d\theta
\]

\[
= 3 \sin \theta \bigg|_{\pi/4}^{\pi/2}
\]

\[
= 3 \left( 1 - \frac{\sqrt{2}}{2} \right).
\]

Thus the area of the region is

\[
A_1 + A_2 = 3(\sqrt{2} + 2).
\]

6. (Test Question) Consider the polar curves \( r = 3 \cos \theta \) and \( r = \sin \theta \). Find all points of intersection of the two curves. Give exact values for the coordinates in \((r, \theta)\). Then find the area of the region that lies inside the graph of \( r = 3 \cos \theta \) and outside of the graph of \( r = \sin \theta \).

**Solution: Intersection pts:** Set the expressions for \( r \) equal to each other, and solve:

\[
\sin \theta = \sqrt{3} \cos \theta
\]

\[
\tan \theta = \sqrt{3},
\]

so \( \theta = \pi/3 + k\pi \), for any integer \( k \).

\[
\theta = \pi/3
\]

Inserting \( \theta = \pi/3 \) into one of the equations for \( r \), we get \( r = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \cdot 3 \), so one intersection point is

\[
\left( \frac{\sqrt{3}}{2}, \frac{\pi}{3} \right).
\]
But were not done, because graphing these two curves in the same plane shows they intersect at two points: one in the first quadrant (which we found above) and the origin. Its acceptable to label this second intersection point (0, 0), as it is that point on the graph of \( r = \sin \theta \). On the graph of \( r = \sqrt{3} \cos \theta \), however, it is actually (0, \( \pi \)). They are the same point with 2 different representation in polar coordinates.\(^a\)

**Area:** By graphing the two curves, we can divide the region in question into a semicircle, which well call A (below the x-axis), and a semicircle with a bite taken out of it, which is B (above the x-axis). Since the diameter of the semicircle runs from \( r = 0 \) to \( r = 3 \), its radius is \( \sqrt{3}/2 \), so

\[
\text{area}(A) = \frac{1}{2} \pi (\frac{\sqrt{3}}{2})^2 = \frac{3\pi}{8} b.
\]

Region B requires an integral. Let \( f(\theta) = \sqrt{3} \cos \theta \) and \( g(\theta) = \sin \theta \); then \( \text{area}(B) \) can be written as

\[
\frac{1}{2} \int_{0}^{\beta} [f(\theta)^2 - g(\theta)^2] \, d\theta.
\]

where \( \beta \) is the angle where \( r = f(\theta) \) and \( r = g(\theta) \) intersect, which we found above to be \( \beta = \frac{\pi}{3} \). Hence

\[
\text{area}(B)
= \frac{1}{2} \int_{0}^{\pi/3} \left[ (\sqrt{3} \cos \theta)^2 - (\sin \theta)^2 \right] \, d\theta
= \frac{1}{2} \int_{0}^{\pi/3} \left[ 3 \cos^2 \theta - \sin^2 \theta \right] \, d\theta
\]

\( ^a \)If you look carefully, a step in the algebra above even hints at this: we had to divide both sides by \( \cos \theta \). But if \( r = 3 \cos \theta = 0 \), then any value of \( \phi \) such that \( r = \sin \phi = 0 \) will put us at the same point, even though \( \theta \neq \phi \), since (0, \( \theta \)) is the same point in the plane as (0, \( \phi \)), for any angles \( \theta \) and \( \phi \).

\( ^b \)You can also find this area by calculating

\[
\int_{-\pi/2}^{0} \frac{1}{2} (\sqrt{3} \cos \theta)^2 d\theta.
\]

\[
= \frac{1}{2} \int_{0}^{\pi/3} \left[ 1 + \cos(2\theta) - \frac{1 - \cos(2\theta)}{2} \right] \, d\theta
= \frac{1}{2} \int_{0}^{\pi/3} \left[ \frac{3 \cos(2\theta)}{2} - \frac{1 - \cos(2\theta)}{2} \right] \, d\theta
= \frac{1}{2} \int_{0}^{\pi/3} \cos(2\theta) \, d\theta
= \frac{1}{2} \left[ \frac{3}{2} \cos(2\theta) - \frac{1 - \cos(2\theta)}{2} \right]_{0}^{\pi/3}
= \frac{1}{2} \left[ \frac{3}{2} \cdot \frac{\pi}{3} - 0 - \frac{3}{4} \right]
= \frac{9\pi}{4}.
\]

7. (Test Question) Find the area of the region bounded by the polar curve \( r = 2 + \sin \theta \).

**Solution:** The equation is of the form \( r = A + B \sin \theta \) with \( A > B \). Hence the graph of the function is a cardioid and one must integrate from 0 to \( 2\pi \) to find its area.

\[
\text{Area} = \frac{1}{2} \int_{0}^{2\pi} (2 + \sin \theta)^2 \, d\theta
= \frac{1}{2} \int_{0}^{2\pi} 4 + 4 \sin \theta + \sin^2 \theta \, d\theta
= \frac{1}{2} \int_{0}^{2\pi} 4 + 4 \sin \theta + \frac{1 - \cos(2\theta)}{2} \, d\theta
= \frac{1}{2} \int_{0}^{2\pi} \frac{9}{2} + 4 \sin \theta + \frac{\cos(2\theta)}{2} \, d\theta
= \frac{1}{2} \left[ \frac{9}{2} \cdot \frac{2\pi}{3} - 4 \cos \left( \frac{\sin(2\theta)}{4} \right) \right]_{0}^{\pi/3}
= \frac{1}{2} \left[ \left( \frac{9}{2} \cdot \frac{2\pi}{3} - 4 \cos(2\pi) - \frac{\sin(2(\pi))}{4} \right) - \left( \frac{9}{2} \cdot 0 - 4 \cos(0) - \frac{\sin(2 \cdot 0)}{4} \right) \right]
= \frac{9\pi}{4}.
\]