Problem 1  [6 points]  Find the length of the curve \( r(t) = (2t, t^2, \ln t) \) for \( 1 \leq t \leq e \).

Solution: Recall that for a parameterized function \( R(t) = (x(t), y(t), z(t)) \), the length of the curve traced out by this function for \( a \leq t \leq b \), is given by

\[
\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt.
\]

In our case \( x(t) = 2t \), \( y(t) = t^2 \), \( z(t) = \ln(t) \), \( a = 1 \), and \( b = e \). After taking a few derivatives, we find that

\[
x'(t) = 2, \quad y'(t) = 2t, \quad \text{and} \quad z'(t) = \frac{1}{t}.
\]

Thus

\[
x'(t)^2 + y'(t)^2 + z'(t)^2 = 2^2 + (2t)^2 + \left( \frac{1}{t} \right)^2 = 4 + 4t^2 + \frac{1}{t^2} = 4t^2 + 4 + \frac{1}{t^2} = \left( 2t + \frac{1}{t} \right)^2.
\]

In light of the formula first mentioned,

\[
\text{Length} = \int_1^e \sqrt{\left( 2t + \frac{1}{t} \right)^2} \, dt.
\]

As \( 2t + \frac{1}{t} > 0 \) for \( t \in [1, e] \),

\[
\int_1^e \sqrt{\left( 2t + \frac{1}{t} \right)^2} \, dt = \int_1^e 2t + \frac{1}{t} \, dt = \left[ t^2 + \ln |t| \right]_1^e = e^2 + 1 - (1 + 0) = e^2.
\]
Problem 1  [4 points]  A projectile with an initial position of \( r(0) = (0, 0, 0) \) is fired with an initial velocity of \( \mathbf{v}(0) = (50, -2, 200) \) (units are in meters per second.) Due to the force of gravity and a wind blowing in the positive \( x \) direction, the acceleration on the object is \( \mathbf{a}(t) = (1, 0, -9.8) \).

a) Find \( \mathbf{v}(t) \).

**Solution:** We take an anti derivative of \( \mathbf{a}(t) \):

\[
\mathbf{v}(t) = (t, 0, -9.8t) + (c_1, c_2, c_3).
\]

Since \( \mathbf{v}(0) = (50, -2, 200) \), \( (c_1, c_2, c_3) = (50, -2, 200) \), and

\[
\mathbf{v}(t) = (t + 50, -2, 200 - 9.8t).
\]

b) Find \( r(t) \).

**Solution:** We take an anti derivative of \( \mathbf{v}(t) \):

\[
r(t) = \left\langle \frac{t^2}{2} + 50t, -2t, 200t - \frac{9.8t^2}{2} \right\rangle + (k_1, k_2, k_3).
\]

Since \( r(0) = (0, 0, 0) \), \( (k_1, k_2, k_3) = (0, 0, 0) \), and

\[
r(t) = \left\langle \frac{t^2}{2} + 50t, -2t, 200t - \frac{9.8t^2}{2} \right\rangle.
\]

c) Find the maximum height of the object (round to two decimal places.)

**Solution:** The time the max height occurs will be when the \( z \) coordinate of \( \mathbf{v}(t) \) is equal to 0. \( 200 - 9.8t = 0 \) has a solution at \( t = \frac{1000}{49} \). The \( z \) coordinate of \( r(\frac{1000}{49}) \) is 2040.82.

d) Find the coordinates of the point where the object hits the ground (round each component to two decimal places.)

**Solution:** The object will hit the ground when the \( z \) coordinate of \( \mathbf{r}(t) \) is equal to 0, which occurs at \( t = 400/9.8 = \frac{2000}{49} \).

\[
r\left(\frac{2000}{49}\right) = (2873.80, -81.63, 0).
\]

e) Find the speed of the object when hits the ground (round to two decimal places.)
f) Assume that the terrain is not level (not the xy-plane), but instead the terrain can be described by the plane $z = 2x + 3y$. Find the time at which the object hits the ground (which is the plane given in this case.) (Round to two decimal places.)

**Solution:** We are interested in identifying intersection points between the parameterized curve and the plane. For a fixed $t$ the point $\mathbf{r}(t) = \left(\frac{t^2}{2} + 50t, -2t, 200t - \frac{9.8t^2}{2}\right)$ will be in the plane if it satisfies the relation defining the plane. In other words it must satisfy

$$200t - \frac{9.8t^2}{2} = 2 \left(\frac{t^2}{2} + 50t\right) + 3 (-2t).$$

This simplifies to

$$106t = 5.9t^2,$$

which has solutions at $t = 0$ and $t = \frac{106}{5.9}$. The time $t = 0$ corresponds to the launch time on this new terrain and the time $t = \frac{106}{5.9}$ will be when the object hits the ground. After rounding to two decimal places we estimate the object will hit the ground at $t = 17.97$ s.