Problem 1  [6 points]  An arch is to be modeled by the function \( y = e^{-3x} + \frac{1}{36}e^{3x} \) on the interval \([-1, 1]\). Find the length of the arch. You do not need to simplify your final answer.

Solution: Recall that the arc length formula of a differentiable curve, \( f(x) \), from \( a \) to \( b \):

\[
\text{Length} = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx
\]

In this case \( a = -1 \), and \( b = 1 \), and \( f(x) = e^{-3x} + \frac{1}{36}e^{3x} \).

\[
f'(x) = -3e^{-3x} + \frac{1}{12}e^{3x}
\]

and

\[
1 + [f'(x)]^2 = 1 + \left(-3e^{-3x} + \frac{1}{12}e^{3x}\right)^2
\]

\[
= 1 + (3e^{-3x})^2 - 2(3e^{-3x}) \cdot \frac{1}{12}e^{3x} + \left(\frac{1}{12}e^{3x}\right)^2
\]

\[
= (3e^{-3x})^2 + 2(3e^{-3x}) \cdot \frac{1}{12}e^{3x} + \left(\frac{1}{12}e^{3x}\right)^2
\]

\[
= \left(3e^{-3x} + \frac{1}{12}e^{3x}\right)^2.
\]

Thus

\[
L = \int_{-1}^{1} \sqrt{(3e^{-3x} + \frac{1}{12}e^{3x})^2}
\]

\[
= \int_{-1}^{1} 4e^{4x} + \frac{1}{16}e^{-4x}
\]

\[
= -e^{-3} + \frac{1}{36}e^{3} \bigg|_{-1}^{1}
\]

\[
= -e^{-3} + \frac{1}{36}e^{3} - \left(-e^{-3}(-1) + \frac{1}{36}e^{3}(-1)\right).
\]
Problem 1  [4 points] Suppose a force of 40 \( N \) is required to stretch and hold a spring 0.8 meters from its equilibrium position.

a) Assuming Hooke’s Law is being obeyed, find the spring constant \( k \).

\[ 40 = k(0.8). \]

So \( k = 50 \)

b) How much work is required to compress the spring 0.3 meters from its equilibrium position?

\[ \text{Solution:} \quad \text{Work} = \int_0^{0.3} 50x \, dx \]
\[ = 50 \left[ \frac{x^2}{2} \right]_0^{0.3} \]
\[ = 25(0.3)^2 J. \]

c) Does it take twice as much work to compress the spring 0.6 meters from equilibrium as it does to compress it 0.3 meters from equilibrium?

\[ \text{Solution:} \quad \text{No.} \quad \text{The work done to compress the spring from} \ 0.6 \ \text{meters to equilibrium is given by} \]
\[ \text{Work} = \int_0^{0.6} 50x \, dx \]
\[ = 50 \left[ \frac{x^2}{2} \right]_0^{0.6} \]
\[ = 25(0.6)^2 J. \]

This value is 4 times that of the work done compressing the spring to 0.3 meters.

d) Does it take twice as much force to compress and hold the spring 0.6 meters from its equilibrium as it does to compress and hold it 0.3 meters from equilibrium?

\[ \text{Solution:} \quad \text{Yes.} \quad \text{By Hooke’s law the force required to keep a spring compressed a distance} \ x \ \text{from equilibrium is} \ F = kx. \ \text{By this formula, doubling} \ x \ \text{will double} \ F. \]