Problem 1  [6 points]  Find the area of the region inside one leaf of \( r = \cos(3\theta) \) (see graph.)

Solution:

Let us begin by calculating the interval of integration. In other words, we will calculate the angles on which the region lies. The region begins at an angle when \( \cos(3\theta) = 0 \) and ends at an angle when \( \cos(3\theta) = 0 \) when \( 3\theta = \pi/2 + k\pi \) for \( k \in \mathbb{N} \). Thus \( \cos(3\theta) = 0 \) when \( \theta = \pi/6 + k/3\pi \) for \( k \in \mathbb{N} \).

Hence the leaf starts at an angle of \(-\pi/6\) and ends at an angle of \(\pi/6\).

There is no inner radius and the outer radius is given by \( r = \cos(3\theta) \). Thus the area is given by:

\[
A = \frac{1}{2} \int_{-\pi/6}^{\pi/6} \cos^2(3\theta) \, d\theta
\]

By symmetry we can integrate from 0 to \(\pi/6\) and multiply by 2. Thus

\[
\text{Area} = \int_0^{\pi/6} \cos^2(3\theta) \, d\theta
\]

\[
= \frac{1}{2} \int_0^{\pi/6} \cos(6\theta) + 1 \, d\theta
\]

\[
= \frac{1}{2} \left[ \frac{\sin(6\theta)}{6} + \theta \right]_0^{\pi/6}
\]

\[
= \frac{1}{2} \left[ \frac{\sin(6 \cdot \pi/6)}{6} + \pi/6 \right] - \frac{1}{2} \left[ \frac{\sin(6 \cdot 0)}{6} + 0 \right]
\]

\[
= \pi/12.
\]
Problem 1  [4 points]  Consider the polar graphs $r = a$ and $r = 2a \sin \theta$ where $a > 0$ is a constant.

a) Find all points of intersection of the two graphs.

**Solution:** The two curves will intersect when $2a \sin \theta = a$. In other words when $\sin \theta = \frac{1}{2}$. This happens for $\theta = \pi/6$ and $\theta = 5\pi/6$.

b) Find the slopes of the lines tangent to the polar curve $r = a$ at the points of intersection found in part (a).

**Solution:** The derivative of a polar function $f(\theta)$ is given by

$$\frac{dy}{dx}(\theta) = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}.$$ 

In our case $f'(\theta) = 0$. Hence

$$\frac{dy}{dx}(\theta) = \frac{a \cos \theta}{-a \sin \theta} = -\cot \theta.$$ 

Furthermore

$$\frac{dy}{dx}(\pi/6) = -\sqrt{3} \quad \text{and} \quad \frac{dy}{dx}(5\pi/6) = \sqrt{3}.$$ 

c) Find the slopes of the lines tangent to the polar curve $r = 2a \sin \theta$ at the points of intersection found in part (a).

**Solution:** We will again use the formula quoted in part (b). In this case $f'(\theta) = 2a \cos \theta$. Thus

$$\frac{dy}{dx}(\theta) = \frac{2a \sin \theta \cos \theta + 2a \cos \theta \sin \theta}{-2a \sin \theta \sin \theta + 2a \cos \theta \cos \theta} = \frac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta}.$$ 

Furthermore

$$\frac{dy}{dx}(\pi/6) = \frac{2 \sin \pi/6 \cos \pi/6}{\cos^2 \pi/6 - \sin^2 \pi/6} = \frac{2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}}{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = \sqrt{3},$$ 

and

$$\frac{dy}{dx}(5\pi/6) = \frac{2 \sin 5\pi/6 \cos 5\pi/6}{\cos^2 5\pi/6 - \sin^2 5\pi/6} = \frac{2 \cdot \frac{1}{2} \cdot -\frac{\sqrt{3}}{2}}{\left(-\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = -\sqrt{3},$$
**d)** Find the area of the region common to (that lies within both) the polar curves $r = a$ and $r = 2a \sin \theta$.

**Solution:** The inner radius of our region is given by $r = 0$ and the outer radius is given first by $r = 2a \sin \theta$ on the interval from 0 to $\pi/6$, then by $r = a$ on the interval $[\pi/6, 5\pi/6]$, and finally by $r = 2a \sin \theta$ on the interval $[5\pi/6, \pi]$.

Thus

\[
\text{Area} = \int_0^{\pi/6} \frac{1}{2} (2a \sin \theta)^2 d\theta + \int_{\pi/6}^{5\pi/6} \frac{1}{2} a^2 d\theta + \int_{5\pi/6}^{\pi} \frac{1}{2} (2a \sin \theta)^2 d\theta.
\]

By symmetry the first and third integrals are the same, and the second integrates to $\frac{\pi}{3} a^2$. Thus all we need to finish is to find the value of the first integral.

\[
\int_0^{\pi/6} \frac{1}{2} (2a \sin \theta)^2 d\theta = \frac{1}{2} 2^2 a^2 \int_0^{\pi/6} 1 - \cos(2\theta) \frac{d\theta}{2} = a^2 \int_0^{\pi/6} 1 - \cos(2\theta) d\theta = a^2 \left[ \theta - \frac{\sin(2\theta)}{2} \right]_0^{\pi/6} = a^2 \left[ \frac{\pi}{6} - \frac{\sin(2 \cdot \pi/6)}{2} \right] + a^2 \left[ 0 - \frac{\sin(2 \cdot 0)}{2} \right] = a^2 \left( \frac{\pi}{6} - \frac{\sqrt{3}}{4} \right)
\]

Thus the area of the common region is $a^2 (2\pi/3 - \sqrt{3}/4)$. 